Examination, 9 June 2022
TMA372 and MMG800

Read this before you start!
I am not able to come to the examination room(s). You can ask for calling me (0317723021) in case of questions.
Aid: Personal pocket calculator.
Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.
I tried to use the same notation as in the lecture.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for me to find the correct one. Justify your answers.
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Use a proper pen and order your answers if possible.
The test has 3 pages and a total of 30 points.
Preliminary grading limits: 3:15-21p, 4:22-28p, 5:29p- (Chalmers) and G:15-26p, VG:27p- (GU)
Valid bonus points will be added to the total score if needed.
You will be informed via Canvas when the exams are corrected.
Good luck!

Some exercises were taken from, or inspired by, materials from *P. Henning*, *J. Hoffman*, *P. Kelly*, and *A. Målqvist*.

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1. Let $\Omega = (0,1)$ and a positive integer N. Consider the mesh defined by the nodes $x_j = j/N$ for $j = 0, 1, \dots, N$ and the corresponding hat functions $\{\varphi_j\}_{j=0}^N$. Let $V_h = \text{span}(\varphi_0, \dots, \varphi_N)$ and $f(x) = \sin(2\pi x)$.

   (a) Sketch the basis function φ_0 and give a formula for the definition of this basis function. (1 p)

   (b) Let $N = 2$ and set $h = 1/N$. Compute the nodal interpolant of f, denoted $\pi_h f \in V_h$. (2 p)

   (c) Let $N = 1$. Compute an approximation of $\int_\Omega f(x)\,\mathrm{d}x$ using the midpoint rule. (1 p)

2. Consider the boundary value problem

$$\begin{cases} -u''(x) = 0 \quad \text{for} \quad x \in (0,2) \\ u'(0) = 1, \quad u(2) = 0. \end{cases}$$

   (a) Integrate the problem twice and give its exact solution. (2 p)

   (b) State the variational formulation to the above problem. (2 p)

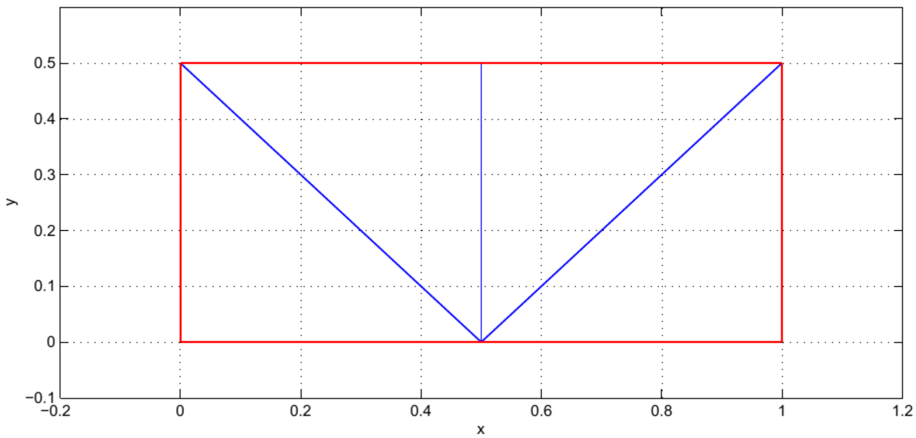

Figure 1: Courtesy from A. Målqvist.

(c) Give the Galerkin (linear) FE problem in the case of one element of length $h = 2$. (2 p)

(d) Finally solve the obtained linear system of equation and provide the FE solution

$$u_h(x) = \zeta_0 \varphi_0(x),$$

where one recalls that $\varphi_0(x) = 1 - x/2$ and ζ_0 is the unknown. (2 p)

3. Let $\Omega = [0,1] \times [0,1]$. Show that the bilinear form

$$a(u,v) = \int_\Omega \nabla u(x,y) \nabla v(x,y) \, \mathrm{d}x \, \mathrm{d}y + \int_\Omega \left(\sqrt{x^2+y^2} + 1 \right) u(x,y) v(x,y) \, \mathrm{d}x \, \mathrm{d}y$$

is $H^1(\Omega)$-elliptic. (2 p)

4. Let H be a Hilbert space. Consider the variational problem

$$\text{Find } u \in H \text{ such that } a(u,v) = \ell(v) \text{ for all } v \in H,$$

where $a(u,v)$ is a nice symmetric bilinear form and ℓ a nice functional. Denote by u_h the corresponding Galerkin finite element approximation u_h of the exact solution u. Show that

$$a(u - u_h, u - u_h) = a(u,u) - a(u_h, u_h). \qquad (2\text{ p})$$

5. Let $\Omega = [0,1] \times [0,0.5]$ with the triangulation in Figure 1. Let the nodes be numbered in the following way $N_1 = (0,0)$, $N_2 = (0.5,0)$, $N_3 = (1,0)$, $N_4 = (1,0.5)$, $N_5 = (0.5,0.5)$, and $N_6 = (0,0.5)$. Derive the point and connectivity matrices describing the location of the nodes and the triangles of the mesh. Note that you can choose any numbering of the triangles. (2 p)

6. Answer the following questions related to the FEM algorithm for $2d$ problems seen in the lecture (you may illustrate your answers with pictures).

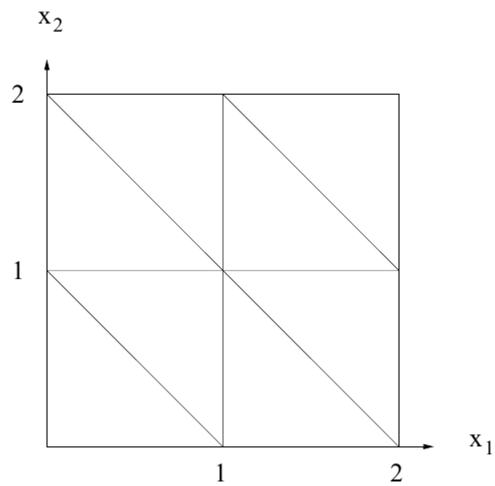

Figure 2: Courtesy from J. Hoffman.

(a) What is a hanging node? (1 p)

(b) Describe the main steps for the assembling procedure of the (mass or stiffness) matrix (you do not have to carry out any computations, just illustrate the idea). (2 p)

(c) Describe how a mapping to a reference triangular element is used to compute element integrals (you do not have to carry out any computations, just illustrate the idea). (2 p)

7. Let Ω be the square from Figure 2. Consider the problem

$$\begin{cases} -\Delta u(x) = 1 & \text{for } x \in \Omega \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $x = (x_1, x_2)$.

(a) Give the finite element approximation of this problem using cG(1) FEM. (3 p)

(b) Compute the 2×2 stiffness matrix where the above homogeneous Dirichlet BC are replaced by the Neumann BC

$$\frac{\partial u}{\partial x_1} = 0,$$

for $x_1 = 2, 0 < x_2 < 2$ (and still homogeneous Dirichlet BC for the rest of the boundary). (4 p)