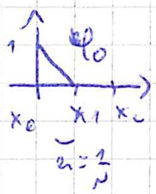
Exam 09.06.22
TMA 372 / MMG 800

1 a)  0,5 $\varphi_0(x) = 1 - Nx$ 0,5

b) $N=2 \Rightarrow x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$ and $f(x_0) = f(x_1) = f(x_2) = 0$ 1

Hence $\pi_h(x) = \sum_{j=0}^{2} f(x_j)\underbrace{\lambda_j(x)}_{\text{Lagrange}} = 0$ 1

c) $\int_0^1 f(x)\,dx \approx (1-0)\, f(\tfrac{1}{2}) = 0$ 1

$\left(\frac{1-1}{24} = 0\right)$

2 a) Integrate $\Rightarrow u(x) = c_0 + c_1 x$, BC $\Rightarrow c_0 = -2, c_1 = 1$ 1

Sol. BVP reads $u(x) = x - 2$ 1

b) Multiply by test fct $v \in V$ and by parts:

$$\underbrace{-u'(2)\,v(2)}_{\text{kill}} + \underbrace{u'(0)\,v(0)}_{\text{keep}} + \int_0^2 u'(x)\,v'(x)\,dx = 0 \quad \forall v \in V$$

with $v(2) = 0$ and $u'(0) = 1$.

(VF) Find $u \in U$ s.t. $(u', v')_{L^2} = 0 - v(0) \quad \forall v \in V$ 1

where $U = \{v \in H^1 \text{ s.t. } v(2) = 0\}$ 1

c) $0 = x_0 \overset{h}{\longmapsto} x_1 = 2 \Rightarrow$ 1 hat fct $\varphi_0(x)$ 1

(FE) Find $u_h = \xi_0 \varphi_0 \in \text{Span}(\varphi_0)$ s.t. $(u_h', \varphi_0')_{L^2} = -\varphi_0(0)$ 1

d) Above gives $\xi_0 (\varphi_0', \varphi_0')_{L^2} = -1$ 1 or $\xi_0 = -2$ hence $u_h(x) = x - 2$ 1
(exact sol!!)

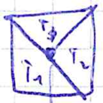
3 $a(u,u) = \int_\Omega \nabla u \nabla u\,dx\,dy + \int_\Omega (\underbrace{\sqrt{x^2+y^2}+1}_{\geq 1})\, u u\,dx\,dy \geq \int \nabla u \nabla u + \int u u = 1 \cdot \|u\|^2_{H^1} \Rightarrow$ 1 1

H^1-elliptic / H^1-coercive

4 $a(u-u_h, u-u_h) \underset{\text{linearity}}{=} a(u, u-u_h) - a(u_h, u-u_h) \underset{\text{sym.}}{=} a(u-u_h, u) - \underbrace{a(u-u_h, u_h)}_{=0 \text{ by GO}} =$ 1

$= a(u,u) - \underbrace{a(u_h,u)}_{\ell(u_h)\ (\text{VF})} = a(u,u) - \ell(u_h) \underset{\text{FE}}{=} a(u,u) - a(u_h,u_h)$ 1

(5) $P = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \\ 1 & 0 \\ 1 & 0.5 \\ 0.5 & 0.5 \\ 0 & 0.5 \end{pmatrix}$ $\qquad T = \left(\begin{array}{ccc|ccc} 1 & 2 & 6 & 1 & 1 & 0 \\ 2 & 5 & 6 & 0 & 0 & 1 \\ 2 & 3 & 4 & 0 & 0 & 1 \\ 2 & 4 & 5 & 1 & 1 & 0 \end{array}\right)$

(6) a) 

hanging

b) (i) One has $S_{ij} = \int_\Omega \nabla\varphi_j \nabla\varphi_i \, dx = \sum_{\ell=1}^{n_t} \int_{K_\ell} \nabla\varphi_i \nabla\varphi_j \, dx$, where S_{ij} → entry stiffness matrix

n_t → # of elements

K_ℓ → ℓ triangle

(ii) Compute element contribution S_ℓ

(iii) Add this contribution at the correct location in the global stiffness matrix S

c) $\int_K g(x,y)\,dx\,dy \overset{\Phi}{=} \int_{\hat{K}} g(\Phi(\xi,\eta)) \underbrace{|J(\xi,\eta)|}_{\text{determinant of Jacobian of } \Phi} d\xi\, d\eta$

nice triangle

Change of variable

ref triangle

$\Phi : (\xi,\eta) \mapsto (x,y)$

isoparametric map

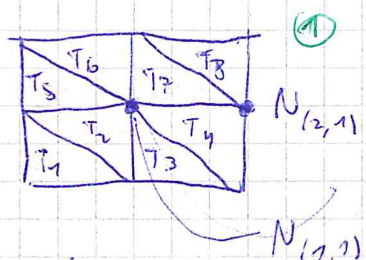
(7) a) VF reads: Find $u \in H_0^1(\Omega)$ s.t. $\int_\Omega \nabla u \nabla v \, dx = \int_\Omega v \, dx \quad \forall v \in H_0^1(\Omega)$

Let $V_h = \{$ cont. p.w. linear fct v on triangulation $\mathcal{T}_h$ with $v = 0$ on $\partial\Omega \} = \operatorname{span}(\varphi_{11})$

FE reads: Find $u_h \in V_h$ s.t. $\int_\Omega \nabla u_h \nabla v_h \, dx = \int_\Omega v_h \, dx \quad \forall v_h \in V_h = \operatorname{span}(\varphi_{1,1})$

→ gives "system" $A\xi = b$

b) Neumann BC ⇒ 2 degrees of freedom: nodes $N_{(1,1)}$ and $N_{(2,1)}$



One has $A = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix}$ with

$A_{11} = \int_\Omega \nabla\varphi_1 \nabla\varphi_1 = \int_{T_2} + \int_{T_3} + \int_{T_4} + \int_{T_5} + \int_{T_6} + \int_{T_7} = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 = 4$

$A_{22} = \int_\Omega \varphi_2 \varphi_2 = \int + \int + \int = 1 + \frac{1}{2} + \frac{1}{2} = 2$, $\quad A_{12} = \int \nabla\varphi_1 \nabla\varphi_2 = \int + \int = -\frac{1}{2} - \frac{1}{2} = -1$