Examination, 25 August 2022
TMA372 and MMG800

Read this before you start!
I'll try to come at ca. 09:30. If I don't manage, you can ask for calling me (0317723021) in case of questions.
Aid: Personal pocket calculator.
Read all questions first and then start to answer the ones you feel most comfortable with. Some parts of an exercise may be independent of the others.
I tried to use the same notation as in the lecture.
Answers may be given in English, French, German or Swedish.
Write down all the details of your computations clearly so that each steps are easy to follow.
Do not randomly display equations and hope for me to find the correct one. Justify your answers.
Write clearly what your solutions are and in the nicest possible form.
Don't forget that you can verify your solution in some cases.
Use a proper pen and order your answers if possible.
The test has 3 pages and a total of 30 points.
Preliminary grading limits: 3:15-21p, 4:22-28p, 5:29p- (Chalmers) and G:15-26p, VG:27p- (GU)
Valid bonus points will be added to the total score if needed.
You will be informed via Canvas when the exams are corrected.
Good luck!

Some exercises were taken from, or inspired by, materials from *M. Bader*, *H. Haddar*, *V. Heuveline*, *J. Hoffman*.

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1. Provide concise answers to these short questions:

   (a) Give a simple example (it does not need to be realistic) of an initial value problem. For a given time step h, apply one step of the explicit/forward Euler method to your IVP. (2 p)

   (b) Is the PDE $\partial_x\partial_y u - \partial_x u = 0$, with $u = u(x, y)$, elliptic, parabolic or hyperbolic? (1 p)

   (c) Is the minimization problem of Poisson's equation in $1d$ equivalent to its variational formulation? (1 p)

   (d) Which function lives in the trial space in the variational form of a PDE? (1 p)

   (e) Use one step of the trapezoidal rule to approximate the area under the function $f(x) = x^2$ between $x = 0$ and $x = 1$. (2 p)

   (f) Describe the main steps for the assembling procedure of the (mass or stiffness) matrix for the FEM algorithm for $2d$ problems seen in the lecture (you do not have to carry out any computations, just illustrate the main idea). (2 p)

2. Let $a < b$ and $p, q\colon (a, b) \to \mathbb{R}$ two piecewise continuous functions with $0 < p_* \leq p(x) \leq p^* < \infty$ and $0 < q_* \leq q(x) \leq q^* < \infty$ for all $x \in (a, b)$. Let $\beta \geq 0$ and $f \in L^2(a, b)$.

Consider the BVP

$$\begin{cases} -(p(x)u'(x))' + q(x)u(x) = f(x) \\ p(b)u'(b) + \beta u(b) = 0 \\ -p(a)u'(a) + \beta u(a) = 0. \end{cases}$$

(a) Write down the variational formulation of the above BVP. (1 p)
Hint: Consider trial and test functions in $H^1(a,b)$.

(b) Denote by $A(\cdot,\cdot)$ and $l(\cdot)$ the bilinear form and the linear functional of the variational formulation. Show that the linear functional $l(\cdot)$ is bounded (or continuous). Show that the bilinear form $A(\cdot,\cdot)$ is coercive (or H^1-elliptic). Finally, use Lax–Milgram to show existence and uniqueness of the solution $u \in H^1(a,b)$ of the variational problem. (4 p)
Hint: You don't need to show continuity of the bilinear form $A(\cdot,\cdot)$, just assume this for answering the last part of the question. If you cannot find $A(\cdot,\cdot)$ or $l(\cdot)$, try to state Lax–Milgram's theorm.

(c) We now consider a cG(1) FE approximation of this problem. Let N be a positive integer and define the mesh $h = \frac{b-a}{N}$ as well as the uniform grid $a = x_0 < x_1 < \ldots < b = x_N$ with $x_j = a + jh$ for $j = 0, 1, \ldots, N$. Define the corresponding FE space V_h and provide a basis for this space. (2 p)

(d) Assume now that $u \in H^2(a,b)$ and denote by u_h the cG(1) approximation. Denote by $\pi_h v$ the piecewise linear interpolant of some $v \in H^1(a,b)$. Use Galerkin's orthogonality to show that

$$A(\pi_h u - u, v) = A(\pi_h u - u_h, v) \quad \text{for all} \quad v \in V_h. \qquad (1\text{ p})$$

(e) Take $v = \pi_h u - u_h \in V_h$ in the above equality and use properties (seen above) of the bilinear form $A(\cdot,\cdot)$ to prove that

$$||\pi_h u - u_h||_{H^1} \leq C_1 \, ||\pi_h u - u||_{H^1}.$$

Using the above, deduce then that

$$||u - u_h||_{H^1} \leq C_2 \, ||\pi_h u - u||_{H^1}. \qquad (2\text{ p})$$

(f) Finally, recall that the interpolation error verifies $||\pi_h u - u||_{H^1} \leq C_3 h$, deduce the error estimate for the cG(1) FEM

$$||u - u_h||_{H^1} \leq Ch. \qquad (1\text{ p})$$

3. Let u_0 and f be given functions. Assume that there are nice solutions, denoted $u_1(x,t)$ and $u_2(x,t)$, to the heat equation

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) \quad 0 < x < 1, 0 < t \leq T \\ u(0,t) = 0, u(1,t) = 0 \quad 0 < t \leq T \\ u(x,0) = u_0(x) \quad 0 < x < 1. \end{cases}$$

Define $w = u_1 - u_2$.

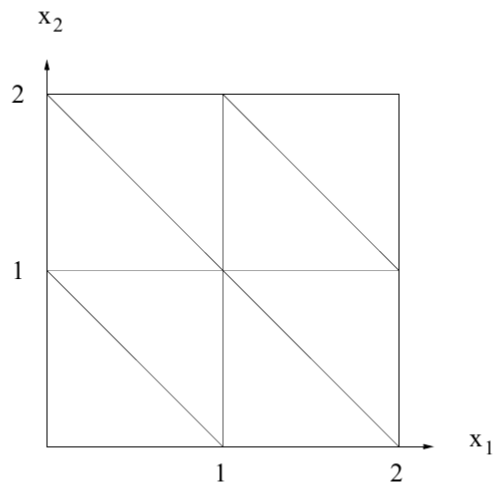

Figure 1: Courtesy from J. Hoffman.

(a) Provide a linear partial differential equation for which w is a solution to. (1 p)

(b) Use the stability estimates (seen in the lecture)

$$\|w(\cdot,t)\|_{L^2} \leq \|\text{initial value of above PDE}\|_{L^2} + \int_0^t \left\|\text{right-hand side of above PDE}\right\|_{L^2} \, \mathrm{d}s$$

to show that $\|w(\cdot,t)\|_{L^2} = 0$ for all t. (1 p)

4. Let $\Omega \subset \mathbb{R}^2$ be the square from Figure 1. Consider the problem

$$\begin{cases} -\Delta u(x) = 1 & \text{for } x \in \Omega \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

where $x = (x_1, x_2)$.

(a) Give the variational formulation of this problem. (1 p)

(b) Give the finite element approximation of this problem using a cG(1) FEM with the mesh from the figure. (2 p)

(c) Compute the 1×1 stiffness matrix, denoted by S, from the above FE problem. (3 p)
Hint: The mesh consists of reference triangles.

(d) Compute the corresponding load vector, denoted by b, from the above FE problem. (1 p)

(e) Solve the "linear system" $Ax = b$ and provide the cG(1) FE approximation of the solution to the above problem. (1 p)