Exam 25.08.22

TMA372, MMG800

① a) $\begin{cases} y'(t) = 3y(t) \\ y(0) = 1 \end{cases}$ (1) Euler reads $y_1 = y_0 + h3y_0 = 1+3h$ (1)

b) Discriminant $d = B^2 - 4AC$ (0,5), where $B=1,\ A=0,\ C=0 \Rightarrow d = 1 > 0 \Rightarrow$ hyperbolic (0,5)

c) Yes, see lecture (1)

d) The sol. to the PDE f. ex. (1)

e) $\int_0^1 f(x)dx \approx \frac{1-0}{2}(f(0)+f(1)) = \frac{1}{2}(0+1) = \frac{1}{2}$ (0,5) (0,5)

f) For stiffness matrix, one has

$S_{ij} = \int_\Omega \nabla\varphi_j \nabla\varphi_i\, dx = \sum_{\ell=1}^{n_t} \int_{K_\ell} \nabla\varphi_i \nabla\varphi_j\, dx$, where S_{ij} entry S

$n_t \to$ # of elements

$K_\ell \to$ triangle ℓ

i) Compute element contribution S_ℓ (1)

ii) Add this contribution at the correct location in the global stiff. matr. S. (1)

② a) Multiply with $v \in H^1$, integrate by parts and get:

$\int_a^b p(x)u'(x)v'(x)dx - [p(x)u'(x)v(x)|_a^b] + \int_a^b q(x)u(x)v(x)dx = \int_a^b f(x)v(x)dx$ (0,5)

(VF) Find $u \in H^1$ s.t. $\int_a^b pu'v'dx + \int_a^b q u v dx + \beta u(b)v(b) + \beta u(a)v(a) = \int_a^b f v\, dx\ \forall v \in H^1$ (0,5)

b) $\ell(v) = \int_a^b f(x)v(x)dx$: $|\ell(v)| \underset{C-S}{\leq} \|f\|_{L^2} \cdot \|v\|_{L^2} \underset{\substack{\text{Def } H^1,\ f \in L^2}}{\leq} C_f \|v\|_{H^1} \to \ell$ bounded (1)

$A(u,v) = \int_a^b p(x)u'(x)v'(x)dx + \int_a^b q(x)u(x)v(x)dx + \beta u(b)v(b) + \beta u(a)v(a)$

$A(u,u) \geq \int_a^b p(x)(u'(x))^2 dx + \int_a^b q(x)(u(x))^2 dx \underset{\substack{\text{def } p,q,H^1}}{\geq} \min(p^*, q^*)\ \|u\|_{H^1}^2$ (1) (1)

$\Rightarrow A$ coercive.

$+ A$ continous/bnd $\overset{LM}{\Rightarrow} \exists!$ sol. of (VF) in H^1 (1)

c) $V_h = \{v : [a,b] \to \mathbb{R},\ v$ cont. pw lin. on grid $T_h\} = \text{span}(\varphi_i)$, where (1) (1)

$\{\varphi_i\}_{i=0}^N$ hat function

d) GO: $A(u-u_h, v_h) = 0\ \forall v_h \in V_h$

this gives $A(-u_h, v_h) = A(-u, v_h)$ or $A(\pi_h u - u, v_h) = A(\pi_h u - u, v_h)$ (1)

c) $A(\Pi_h u - u, \Pi_h u - u_h) = A(\Pi_h u - u_h, \Pi_h u - u_h)$

we: $\|\Pi_h u - u_h\|^2_{H^1} \underset{\text{A coercive}}{\leq} C \cdot A(\Pi_h u - u_h, \Pi_h u - u_h) = A(\Pi_h u - u, \Pi_h u - u_h) \underset{\text{continuity}}{\leq} \|\Pi_h u - u\|_{H^1} \, \cancel{\|\Pi_h u - u_h\|_{H^1}}$ 1

ow: $\|u - u_h\|_{H^1} \overset{\Delta}{\leq} \|u - \Pi_h u\|_{H^1} + \|\Pi_h u - u_h\|_{H^1} \overset{\text{above}}{\leq} C \|u - \Pi_h u\|_{H^1}$ 1

) $\|u - u_h\|_{H^1} \underset{\text{above}}{\leq} C \cdot \|\Pi_h u - u\|_{H^1} \underset{\text{interpolation error}}{\leq} C \cdot h$ 1

) a) ~~By~~ By linearity, w solves the linear PDE

$$\begin{cases} w_t - w_{xx} = 0 & \text{on } (0,1) \\ w(0,t) = 0 = w(1,t) \\ w(x,0) = 0 \end{cases}$$ 1

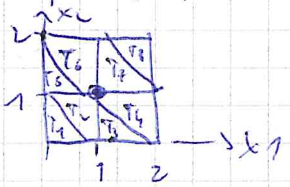
b) $\|w(\cdot,t)\|_{L^2} \leq 0 + \int_0^t 0 \, ds = 0$ 1

) a) (VF) Find $u \in H_0^1$ s.t. $\int_\Omega \nabla u \nabla v \, dx = \int_\Omega 1 \cdot v \, dx \quad \forall v \in H_0^1$ (see lecture) 1

) (FE) Find $u_h \in V_h^0$ s.t. $\int_\Omega \nabla u_h \nabla v_h \, dx = \int_\Omega 1 \, v_h \, dx \quad \forall v_h \in V_h^0$, 1

where $V_h^0 = \{ v_h : \Omega \to \mathbb{R} \text{ cont. and pw linear on grid } T_h \text{ with } v_h = 0 \text{ on } \partial\Omega \} =$

$= \text{span}(\varphi_1)$, where φ_1 is the hat function on node $(1,1)$. 1



) Write $u_h(x) = \xi_1 \varphi_1(x)$, where ξ_1 is unknown. Take $v_h = \varphi_1$ and get

FE) $\underbrace{\xi_1 \int_\Omega \nabla\varphi_1 \nabla\varphi_1 \, dx}_{S} = \underbrace{\int_\Omega \varphi_1 \, dx}_{b}$ 1

$S = \int_{T_2} + \int_{T_3} + \int_{T_4} + \int_{T_5} + \int_{T_6} + \int_{T_7}$ 0,5

$\int_{T_2} = \int_{T_7} = \int_{\Delta} \nabla(1 - x_1 - x_2) \nabla(1 - x_1 - x_2) \, dx = \int_{\Delta} (-1,-1)(-1,-1) dx = 2 \,\text{area}(\Delta) = 2 \cdot \frac{1}{2} = 1$ 0,5

$\int_{T_3} = \int_{T_4} = \int_{T_5} = \int_{T_6} = \int_{\Delta} \nabla(x_2) \nabla(x_2) \, dx = \int_{\Delta} 1 \, dx = \frac{1}{2}$ 0,5

$\Rightarrow S = 4 \cdot \frac{1}{2} + 2 \cdot 1 = 4$ 0,5

) $b = \int_\Omega \varphi_1 \, dx = \text{volume under } \varphi_1 = 6 \cdot \frac{\frac{1}{2} \cdot 1}{3} = 1$ (pyramid with triangular base) 1

e) $\xi_1 = \frac{1}{4} \Rightarrow u_h(x) = \frac{1}{4} \varphi_1(x)$ 1