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Exercises on linear ODE with periodic coefficients.

1. Find the characteristic (Floquet) multiplicator for the scalar linear equation with periodic coefficient: **(4p)**

$$x' = (a + \sin^2 t)x$$

Find also those values of the parameter a that imply that all solutions tend to zero with $t \to +\infty$.

2. Calculate monodromy matrix and Floquet exponents for the 2-dim system

$$x'(t) = a(t)Ax$$

where $a(t)$ is a scalar periodic function with period T and A is a constant real 2×2 matrix. Discuss conditions implying that all solutions tend to zero or stay bounded with $t \to +\infty$.

Hint: make a change of time variable $t \to \tau = \int_{t_0}^{t} a(s)ds$.

3. Compute the monodromy matrix for the system with the following periodic matrix $A(t)$ with period 1.

$$A(t) = \begin{cases} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} = A_1, & 0 \leq t < 1/2 \\ \\ \begin{bmatrix} \alpha & 0 \\ 1 & \alpha \end{bmatrix} = A_2, & 1/2 \leq t < 1 \end{cases}$$

Hint: combine explicit formulas for fundamental matrices on subintervals where $A(t)$ is a constant matrix and the Chapmen-Kolmogorov relation.

4. Consider the following linear system of ODE with periodic coefficients: $\dfrac{d\overrightarrow{r}(t)}{dt} = A(t)\overrightarrow{r}(t)$, with matrix $A(t) = (a + \sin^2(t))\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.
   Use Floquet theory to find for which real constants a its solutions are bounded. *Hint:* make a change of the time variable as in Exercise 2 to find a monodromy matrix.

5. Exercise 2.21. p.58. Consider the Hill equation $y'' + a(t)y = 0$; $a(t+p) = a(t)$.with periodic $a(t)$ with period $p = 1$ having the form:

$$a(t) = \begin{cases} \omega^2, & m \le t < m + \tau \\ 0, & m + \tau \le t < m + 1 \end{cases}$$

Here $\tau \in (0,1)$, $\omega = \pi/\tau$.

The vector form of the Hill equation is:

$$\begin{aligned} x' &= A(t)x \\ A(t) &= \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix} \end{aligned}$$

Consider the transfer matrix solution $\Phi(t,0)$ and show that its first column $\Phi_1(t,0)$ is periodic with period 2, and it's second column $\Phi_2(t,0)$ is unbounded with the first element equal to $(-1)^n n(1-\tau)$.

Some solutions

1. Find the characteristic multiplicator for the scalar linear equation with periodic coefficient: **(4p)**

$$x' = (a + \sin^2 t)x$$

The characteristic multiplicator is eigenvalue of the monodromy matrix denoted by $\Phi(p, 0)$ in the course book, where p is the period of the right hand side in the equation. One builds a monodromy matrix (it will be a number in our case with one scalar equation) of solutions to initial value problems with initial data $x(0)$ that are standard basis vectors in R^n calculated in the time point T - equal to the period of the right hand side.In our case we have just one scalar equation, so the monodromy matrix will be a number. We find the value of the solution to I.V.P. to the given equation with initial data $x(0) = 1$ at the time $t = \pi$ that is a period of the right hand side in our case. The equation is linear, so the solution is found with help of a primitive function of the coefficient:

1. $P(t) = \int_0^t (a + \sin^2 s)ds = \frac{1}{2}t + at - \frac{1}{4}\sin 2t.$
   $x(t) = \exp(P(t))x(0) = \exp\left(\frac{1}{2}t + at - \frac{1}{4}\sin 2t\right)x(0).$
   The monodromy "matrix" in our case is the value of the solution $x(t)$ in $t = \pi$ such that $x(0) = 1$.
   $\Phi(\pi, 0) = x(\pi) = \exp\left(\frac{1}{2}\pi + a\pi\right) = \exp\left(\pi(1/2 + a)\right).$
   The characteristic multiplicator is the same number: $\exp\left(\pi(1/2 + a)\right)$.
   Solutions will tend to zero in the case $a < -1/2$, that makes $\exp\left(\pi(1/2 + a)\right) < 1$.

3. Compute the monodromy matrix for the system $x'(t) = A(t)x(t)$ with the following periodic matrix $A(t)$ with period 1.

$$A(t) = \begin{cases} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} = A_1, & 0 \le t < 1/2 \\ \\ \begin{bmatrix} \alpha & 0 \\ 1 & \alpha \end{bmatrix} = A_2, & 1/2 \le t < 1 \end{cases}$$

Solution:

The monodromy matrix $\Phi(p,0) = \Phi(1,0)$ is expressed as (using Chapman-Kolmogorov)

$$\begin{aligned} \Phi(1,0) &= \Phi(1,1/2)\Phi(1/2,0) \\ &= \exp((1-1/2)A_2)\exp((1/2)\,A_1) \\ &\quad \exp((1/2)A_2)\exp((1/2)\,A_1) \end{aligned}$$

Here $\exp(tA_1) = \exp(\alpha t)\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, $\exp(tA_2) = \exp(\alpha t)\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$

We derive an explicit expression for $\Phi(1,0)$ $\Phi(1,0) = \exp(\frac{1}{2}\alpha + \frac{1}{2}\alpha)\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$

$= \exp(\alpha)\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix}$,

det $\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix} = 1$; $Tr\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix} = 2.25$.

characteristic polynomial $p(\lambda) = \lambda^2 - \frac{9}{4}\lambda + 1$

eigenvalues: $\lambda_1 = \frac{9}{8} - \sqrt{\left(\frac{9}{8}\right)^2 - 1} = \frac{9}{8} - \frac{1}{8}\sqrt{17} > 0$, $\lambda_2 = \frac{1}{8}\sqrt{17} + \frac{9}{8} > 0$ and are simple.

Find conditions on α such that all solutions will be bounded

The condition for boundedness of all solutions is $\exp(\alpha)\,|\lambda_2| \le 1$ or $\exp(\alpha)\frac{1}{8}\left(\sqrt{17}+9\right) \le 1$ because λ_2 is larger in absolute value.

It can be reformulated by taking logarithm of left and right hand sides as $\alpha \le \ln(8) - \ln(\sqrt{17}+9) \approx -0.494\,93$.

All solutions will tend to zero if and only if the strict inequality is valid $\alpha < \ln(8) - \ln(\sqrt{17}+9) \approx -0.494\,93$

4. Consider the following linear system of ODE with periodic coefficients: $\frac{d\overrightarrow{r}(t)}{dt} = A(t)\overrightarrow{r}(t)$, with matrix $A(t) = (a + \sin^2(t))\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Use Floquet theory to find for which real constants a its solutions are bounded.

Hint: make a change of the time variable to find a monodromy matrix. **(4p)**

Solution. The period of the coefficients is $p = \pi$.

Consider the equation in the form

$$\frac{1}{(a + \sin^2(t))}\frac{d\overrightarrow{r}(t)}{dt} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\overrightarrow{r}$$

Introduce a new time variable $\tau(t) = \int_0^t (a + \sin^2(s))ds$. The change of variables in time differentiation will be

$$\frac{d}{dt} = \frac{d\tau(t)}{dt}\frac{d}{d\tau} = \frac{d\left(\int_0^t (a + \sin^2(s))ds\right)}{dt}\frac{d}{d\tau} = (a + \sin^2(t))\frac{d}{d\tau}$$

$$\tau(t) = \int_0^t a + \sin^2(s)ds = at + \frac{1}{2}t - \frac{1}{4}\sin 2t$$

Therefore

$$\frac{1}{(a + \sin^2(t))}\frac{d\overrightarrow{r}(t)}{dt} = \frac{d\overrightarrow{r}(\tau)}{d\tau} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\overrightarrow{r} = B\overrightarrow{r}$$

and we have got a linear system of ODEs with constant coefficients in terms of the τ variable and can solve it exactly. Its transfer matrix is $\exp(\tau B)$ and

$$\overrightarrow{r}(\tau) = \exp(\tau B)r(0)$$

with $\exp(0 \cdot B) = I$.

The transfer matrix for the original system is

$$\Phi(t, 0) = \exp(\tau(t)B)$$

with

$$\tau(t) = at + \frac{1}{2}t - \frac{1}{4}\sin 2t$$

and we observe that $\exp(B\tau(0)) = I$. The monodromy matrix of the original system will be $\Phi(\pi, 0)$ because the period of the coefficients is $p = \pi$.

$$\Phi(\pi, 0) = \exp(\tau(\pi)B)$$

The characteristic polynomial for the matrix $B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \lambda^2 - 3\lambda + 1$

Eigenvalues of the matrix B are $\lambda_1 = \frac{3}{2} - \frac{1}{2}\sqrt{5}$ and $\lambda_2 = \frac{1}{2}\sqrt{5} + \frac{3}{2}$ - both positive. Floquet multipliers are $\exp(\lambda_1\tau(\pi))$ and $\exp(\lambda_2\tau(\pi))$ and are semisimple. Floquet exponents are evidently $\frac{1}{\pi}(\lambda_1\tau(\pi))$ and $\frac{1}{\pi}(\lambda_2\tau(\pi))$.

We must have $\tau(\pi) \leq 0$ to have the both Floquet exponents non-positive and correspondingly to have Floquet multipliers not larger than 1.

It will imply by the Floquet theorem that solutions to the given system of ODE will be bounded because $\lambda_1\tau(\pi)$ and $\lambda_2\tau(\pi)$ are different (not multiple). Checking the values of the integral $\tau(\pi) = \int_0^{\pi}(a + \sin^2(s))ds = a\pi + \frac{1}{2}\pi - \frac{1}{4}\sin 2\pi$ we observe that to have $\tau(2\pi) \leq 0$, a must satisfy the inequality $a \leq -1/2$. The same idea would in fact work for any function instead of $(a + \sin^2(s))$ in the definition of $A(t)$. See Exercise 2.

5. Exercise 2.21. p.58.

Consider the Hill equation $y'' + a(t)y = 0$; $a(t + p) = a(t)$.with periodic $a(t)$ with period $p = 1$. The vector form with $x_1(t) = y(t)$, $x_2(t) = y'(t)$ of the equation is:

$$\begin{aligned} x' &= A(t)x \\ A(t) &= \begin{bmatrix} 0 & 1 \\ -a(t) & 0 \end{bmatrix} \end{aligned}$$

We choose $a(t)$ as a piecewise constat periodic function:

$$a(t) = \begin{cases} \omega^2, & m \leq t < m + \tau \\ 0, & m + \tau \leq t < m + 1 \end{cases}$$

Here $\tau \in (0, 1)$, $\omega = \pi/\tau$.

Consider the transfer matrix solution $\Phi(t, 0)$ and show that its first column $\Phi_1(t, 0)$ is periodic with period 2, and it's second column $\Phi_2(t, 0)$ is unbounded with the it's element equal to $(-1)^n n(1 - \tau)$.

Solution. The monodromy matrix has the followinf structure:

$$\Phi(1, 0) = \Phi(1, \tau)\Phi(\tau, 0) = \exp((1 - \tau)A_2)\exp(\tau A_1)$$

where according to the definition of $A(t)$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} = A(t)\ ,\ t \in (0,\tau)$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = A(t), \quad t \in (\tau, 1)$$

Eigenvectors to A_1 are: $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \left\{ \begin{bmatrix} -\frac{i}{\omega} \\ 1 \end{bmatrix} \right\} \leftrightarrow i\omega,$
$\left\{ \begin{bmatrix} \frac{i}{\omega} \\ 1 \end{bmatrix} \right\} \leftrightarrow -i\omega.$
Check the first of eigenvectors:
$\begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = i\omega \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\begin{aligned} v_2 &= i\omega v_1 \\ -\omega^2 v_1 &= i\omega v_2 \end{aligned}$$

$x_*(t) = \left(\begin{bmatrix} -\frac{i}{\omega} \\ 1 \end{bmatrix} \exp(i\omega t) \right) = \begin{bmatrix} -\frac{i}{\omega} \\ 1 \end{bmatrix} (\cos(\omega t) + i\sin(\omega t)) = \begin{bmatrix} -\frac{i}{\omega}(\cos t\omega + i\sin t\omega) \\ \cos t\omega + i\sin t\omega \end{bmatrix}$
; $\operatorname{Re} x_*(t) = \begin{bmatrix} \frac{1}{\omega}(\sin t\omega) \\ \cos t\omega \end{bmatrix}$; $\operatorname{Im} x_*(t) = \begin{bmatrix} -\frac{1}{\omega}\cos t\omega \\ \sin t\omega \end{bmatrix}$

We like to build using these two linearly independent solutions, one solution with initial data $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and one solution with initial data $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It is easy to see that the following solutions satisfy these initial conditions and can be collected into the transfer matrix:

$$\begin{aligned} \Phi(t,0) &= [-\omega \operatorname{Im} x_*(t), \operatorname{Re} x_*(t)] = \begin{bmatrix} \cos t\omega & \frac{1}{\omega}(\sin t\omega) \\ -\omega \sin t\omega & \cos t\omega \end{bmatrix} \\ \Phi(\tau,0) &= \begin{bmatrix} \cos \tau\omega & \frac{1}{\omega}(\sin \tau\omega) \\ -\omega \sin \tau\omega & \cos \tau\omega \end{bmatrix} \end{aligned}$$

We will calculate $\Phi(t,\tau)$ for $t \in (\tau, 1]$.

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

A_2 is a Joirdan block with eigenalue $\lambda = 0$.

Then $\Phi(t,\tau) = \exp\left((t-\tau)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix}$ according to formulas for a Jordan block.

Then $\Phi(1,\tau) = \begin{bmatrix} 1 & 1-\tau \\ 0 & 1 \end{bmatrix}$;

The monodromy matrix is calculated as:

$$
\begin{aligned}
\Phi(1,0) &= \Phi(1,\tau)\Phi(\tau,0) = \begin{bmatrix} 1 & 1-\tau \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\tau\omega & \frac{1}{\omega}(\sin\tau\omega) \\ -\omega\sin\tau\omega & \cos\tau\omega \end{bmatrix} \\
&= \begin{bmatrix} \cos\tau\omega - \omega(\sin\tau\omega)(1-\tau) & \frac{1}{\omega}\sin\tau\omega + (\cos\tau\omega)(1-\tau) \\ -\omega\sin\tau\omega & \cos\tau\omega \end{bmatrix}
\end{aligned}
$$

If $\omega = \pi/\tau$, then the monodromy matrix is

$$
\begin{aligned}
\Phi(1,0) &= \begin{bmatrix} \cos\pi - \omega(\sin\pi)(1-\tau) & \frac{1}{\omega}\sin\pi + (\cos\pi)(1-\tau) \\ -\omega\sin\pi & \cos\pi \end{bmatrix} \\
&= \begin{bmatrix} -1 & -(1-\tau) \\ 0 & -1 \end{bmatrix}
\end{aligned}
$$

Eigenvalues of this triangular monodromy matrix are both equal to $\lambda_{1,2} = -1$.

Checking the matrix $\Phi(1,0) - (-1)I = \begin{bmatrix} 0 & -(1-\tau) \\ 0 & 0 \end{bmatrix}$ we find only one linearly independent eigenvector to $\Phi(1,0)$ is $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

1) Therefore there must exist unbounded solutions because the multiple $\lambda_{1,2} = -1$ is not semisimple. (!!!!)

2) Therefore $(\lambda_{1,2})^2 = 1$. It implies by a Corollary preious time that the solution with initial data equal to the corresponding eigenvector e_1 has the period $2p = 2$ that is double period of the system. In this particular case the period of coefficients is $p = 1$.

$$
\begin{aligned}
A_1 v &= \lambda v, \quad v \text{ - an eigenvector} \\
x_*(t) &= \exp(t\lambda)v \quad \text{is a solution to} \\
x' &= A_1 x
\end{aligned}
$$

This solution is the first column in $\Phi(t,0)$, because the corresponding eigenvector $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ - is the initial condition for the first column in $\Phi(t,0)$.

In time points $t = pn = n$ the second column in $\Phi(t,0)$ is equal to the second column in $\Phi(1,0)^n$ -that is the n - th power of the monodromy matrix that coinsides with $\Phi(t,0)$ for t equal to integer number of periods.

$\Phi(1,0)^2 = \begin{bmatrix} -1 & -(1-\tau) \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -(1-\tau) \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2\tau+2 \\ 0 & 1 \end{bmatrix}$

$\Phi(1,0)^3 = \begin{bmatrix} -1 & -(1-\tau) \\ 0 & -1 \end{bmatrix}^3 = \begin{bmatrix} -1 & 3\tau-3 \\ 0 & -1 \end{bmatrix}$

$\Phi(1,0)^4 = \begin{bmatrix} -1 & -(1-\tau) \\ 0 & -1 \end{bmatrix}^4 = \begin{bmatrix} 1 & -4\tau+4 \\ 0 & 1 \end{bmatrix}$

We observe that $\Phi(1,0)^n = \begin{bmatrix} 1 & (-1)^n n\,(1-\tau) \\ 0 & (-1)^n \end{bmatrix}$ and the exercise is finished.

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