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**Exercises on general linear ODE**

1. Show that $(A(t)B(t))' = A'(t)B(t) + A(t)B'(t)$ for $n \times n$ matrices $A(t)$ and $B(t)$ with differentiable elements.

2. Show that $\det(\exp(A)) = \exp(\mathrm{tr}A)$ for any constant matrix $A$.

3. If $t \longmapsto \Psi(t)$ is a fundamental matrix solution for the system $x' = A(t)x$, $x \in \mathbb{R}^n$. It means that $\Psi'(t) = A(t)\Psi(t)$.

Then the matrix valued function $\Phi(t,\tau) = \Psi(t)\Psi^{-1}(\tau)$ is called the transition matrix function: it is a fundamental matrix solution with respect to the variable $t$ for each $\tau$ such that $\Phi(\tau,\tau) = I$. It implies that the solution $x(t)$ to I.V.P.

$$x' = A(t)x, \quad x(\tau) = \xi$$

with initial data $\xi$ at the time $\tau$ is given by the expression:

$$x(t) = \Phi(t,\tau)\xi$$

The matrix $\Phi(t,\tau)$ satisfies Chapman-Kolmogorov identities:

$$\Phi(t,s)\Phi(s,\tau) = \Phi(t,\tau)$$

(semigroup property) and

$$\Phi^{-1}(t,s) = \Phi(s,t), \qquad \frac{\partial \Phi(t,s)}{\partial s} = -\Phi(t,s)A(s)$$

Prove these statements.

4. Calculate the transition matrix function $\Phi(t,s)$ for the system of equations

$$\begin{cases} x_1' = t\,x_1 \\ x_2' = x_1 + t\,x_2 \end{cases}$$

5. Calculate the transition matrix function $\Phi(t,s)$ for the system of equations

$$\begin{cases} x_1' = x_1 + tx_2 \\ x_2' = 2x_2 \end{cases}$$

6. Suppose that every solution of $x' = A(t)x$ is bounded for $t \geq 0$ and let $\Psi(t)$ be a fundamental matrix solution. Prove that $\Psi^{-1}(t)$ is bounded for $t \geq 0$ if and only if the function $t \rightarrow \int_0^t \mathrm{tr}A(s)ds$ is bounded from below. Hint: The inverse of a matrix is the adjugate of the matrix divided by its determinant. See: http://en.wikipedia.org/wiki/Adjugate_matrix

7. Suppose that the linear system $x' = A(t)x$ is defined on an open interval containing the origin whose right-hand end point is $\omega \leq \infty$ and the norm of every solution has a finite limit as $t \rightarrow \omega$.

Show that there is a solution converging to zero as $t \rightarrow \omega$ if and only if $\int_0^\omega \mathrm{tr}A(s)ds = -\infty$. **Hint**: Use Abels formula and the fact that a matrix has a nontrivial kernel if and only if its determinant is zero.

7a. Show that if $\liminf_{t\to+\infty}\int_{t_0}^{t} tr(A(s))ds = +\infty$ then the equation $x' = A(t)x$ has an unbounded solution. Hint: use Abel's formula.

8. Let $A$ be an invertible constant matrix. Show that the only invariant lines for the linear system $x' = Ax, \ \ x \in R^2$ are the lines $ax_1 + bx_2 = 0$ where $[-b, a]^T$ is an eigenvector to $A$.

9. Show that for arbitrary $n \times n$ matrix $A$ the relation $\det(I + \varepsilon A + O(\varepsilon^2)) = 1 + \varepsilon \operatorname{tr}(A) + O(\varepsilon^2)$

10. Consider the flow $\phi(t, x)$ corresponding to the autonomous equation $y' = f(y)$, $y \in \mathbb{R}^n$ mapping the domain $\Omega$ to the domain as $\Omega_t = \{y = \phi(t, x), x \in \Omega\}$ where $y$ is the solution to the ODE $y' = f(y)$ with initial data $y(0) = x \in \Omega$.

Show that $\frac{d}{dt}(Vol(\Omega_t)) = \int_{\Omega_t} \operatorname{div}(f)\, dy$. **Hint:** use the result of Ex.9.

11. Show directly that the area of a unit disk is preserved when it is transformed forward to 2 time units by the flow, corresponding to the system $x' = y$, $y' = x$. **Hint**: consider the system in new variables $x + y$ and $x - y$.

## Solutions.

### Solution to 3.

- $\Phi(t,s)\Phi(s,\tau) = \Psi(t)\Psi^{-1}(s)\Psi(s)\Psi^{-1}(\tau) = \Psi(t)\Psi^{-1}(\tau) = \Phi(t,\tau)$.
- $\Phi^{-1}(t,s) = \left(\Psi(t)\Psi^{-1}(s)\right)^{-1} = \left(\Psi^{-1}(s)\right)^{-1}\left(\Psi(t)\right)^{-1} = \Psi(s)\Psi^{-1}(t) = \Phi(s,t)$,
- $\frac{\partial\Phi(t,s)}{\partial s} = -\Phi(t,s)A(s)$

  Use the relation: $\frac{d}{ds}\left(\Psi^{-1}(s)\right) = -\Psi^{-1}(s)\frac{d}{ds}\left(\Psi(s)\right)\Psi^{-1}(s)$

  $\frac{\partial\Phi(t,s)}{\partial s} = \frac{\partial\left(\Phi^{-1}(s,t)\right)}{\partial s} = \left(-\Phi^{-1}(s,t)\frac{\partial}{\partial s}\left(\Phi(s,t)\right)\Phi^{-1}(s,t)\right) = -\Phi^{-1}(s,t)A\,\Phi(s,t)\Phi^{-1}(s,t) = -\Phi^{-1}(s,t)A = -\Phi(t,s)A$

### Solution to 4.

Calculate the transition matrix function $\Phi(t,s)$ for the system of equations

$$\begin{cases} x_1' = t\,x_1 \\ x_2' = x_1 + t\,x_2 \end{cases}$$

$$x' = A(t)x$$

$x(t) = \Phi(t,\tau)\xi$

Here the matrix $A(t)$ is triangular.

Solution to the scalar linear equation $x' = p(t)x + g(t)$ with initial data $x(\tau) = x_0$ is calculated using the primitive function $\mathbb{P}(t,\tau)$ of $p(t)$.

$$\begin{aligned} x' &= p(t)x + g(t) \\ \mathbb{P}(t,\tau) &= \int_\tau^t p(s)ds \\ x(t) &= \exp\{\mathbb{P}(t,\tau)\}\,x_0 + \int_\tau^t \exp\{\mathbb{P}(t,s)\}\,g(s)ds \\ x(\tau) &= x_0 \end{aligned}$$

A derivation of this formula using the integrating factor idea follows.

$$\begin{aligned} x' &= p(t)x + g(t), \quad x_0 = x(\tau) \\ \mathbb{P}(t,\tau) &= \int_\tau^t p(s)ds \\ \exp\{-\mathbb{P}(t,\tau)\}\,x' &= \exp\{-\mathbb{P}(t,\tau)\}\,p(t)x + \exp\{-\mathbb{P}(t,\tau)\}\,g(t) \\ \exp\{-\mathbb{P}(t,\tau)\}\,x' - p(t)\exp\{-\mathbb{P}(t,\tau)\}\,x &= \exp\{-\mathbb{P}(t,\tau)\}\,g(t) \\ \exp\{-\mathbb{P}(t,\tau)\}\,x' + \left(\exp\{-\mathbb{P}(t,\tau)\}\right)'x &= \exp\{-\mathbb{P}(t,\tau)\}\,g(t) \\ \left[\exp\{-\mathbb{P}(t,\tau)\}\,x\right]' &= \exp\{-\mathbb{P}(t,\tau)\}\,g(t) \\ \int_\tau^t \left[\exp\{-\mathbb{P}(s,\tau)\}\,x(s)\right]' ds &= \int_\tau^t \exp\{-\mathbb{P}(s,\tau)\}\,g(s)ds \\ \exp\{-\mathbb{P}(t,\tau)\}\,x(t) - \exp\{-\mathbb{P}(\tau,\tau)\}\,x_0 &= \int_\tau^t \exp\{-\mathbb{P}(s,\tau)\}\,g(s)ds \\ \exp\{-\mathbb{P}(t,\tau)\}\,x(t) - \exp\{0\}\,x_0 &= \int_\tau^t \exp\{-\mathbb{P}(s,\tau)\}\,g(s)ds \end{aligned}$$

$$
\begin{aligned}
x(t) &= \exp\left\{\mathbb{P}(t,\tau)\right\} x_0 + \int_\tau^t \exp\left\{\mathbb{P}(t,\tau)\right\} \exp\left\{-\mathbb{P}(s,\tau)\right\} g(s)ds \\
x(t) &= \exp\left\{\mathbb{P}(t,\tau)\right\} x_0 + \int_\tau^t \exp\left\{\mathbb{P}(t,\tau) - \mathbb{P}(s,\tau)\right\} g(s)ds \\
\mathbb{P}(t,\tau) - \mathbb{P}(s,\tau) &= \int_\tau^t p(z)dz - \int_\tau^s p(z)dz = \int_\tau^t p(z)dz + \int_s^\tau p(z)dz = \\
\int_s^t p(z)dz &= \mathbb{P}(t,s) \\
x(t) &= \exp\left\{\mathbb{P}(t,\tau)\right\} x_0 + \int_\tau^t \exp\left\{\mathbb{P}(t,s)\right\} g(s)ds; \\
x(\tau) &= x_0
\end{aligned}
$$

The system of ODE above has triangular matrix and can be solved recursively starting from the first equation.

The fundamental matrix $\Phi(t,\tau)$ satifies the same equation, namely

$$
\begin{aligned}
\frac{d}{dt}\Phi(t,\tau) &= A\Phi(t,\tau) \\
\Phi(\tau,\tau) &= I
\end{aligned}
$$

$\Phi(t,\tau)$ has columns $\pi_1(t,\tau)$ and $\pi_2(t,\tau)$ that at the time $t=\tau$ have initial values $[1,0]^T$ and $[0,1]$, because $\Phi(\tau,\tau) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

In the equation $x_1' = t\,x_1$ the coefficient $p(t) = t$, therefore $\mathbb{P}(t,\tau) = \int_\tau^t s\,ds = \left(\frac{1}{2}s^2\right)\Big|_\tau^t = \frac{1}{2}\left(t^2 - \tau^2\right)$ and the solution $x_1(t) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_1(\tau)$.

The second equation $x_2' = t\,x_2 + x_1$ is similar but inhomogeneous:

$$x_2(t) = \exp(\mathbb{P}(t,t_0))x_2(t_0) + \int_{t_0}^t \exp(\mathbb{P}(t,s))x_1(s)ds.$$

Substituting $\mathbb{P}(t,\tau) = \frac{1}{2}\left(t^2-\tau^2\right)$ we conclude that $x_2(t) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau) + \int_\tau^t \exp(\frac{1}{2}\left(t^2-s^2\right))\exp(\frac{1}{2}\left(s^2 -\right.$ $\exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau) + \int_\tau^t \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_1(\tau)ds$

And

$$x_2(t) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau) + \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_1(\tau)(t-\tau).$$

The fundamental matrix solution $\Phi(t,\tau)$ has columns that are solutions to $x' = A(t)x$ with initial data - that are columns in the unit matrix: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

Taking $x_1(\tau) = 1$ and $x_2(\tau) = 0$ we get $x_1(t) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))$ with $x_2(t) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))(t-\tau)$

Taking $x_1(\tau) = 0$ and $x_2(\tau) = 1$ we get $x_1(t) = 0$ with $x_2(t) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))$ and the fundamental matrix solution in the form

$$\Phi(t,\tau) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))\begin{bmatrix} 1 & 0 \\ t-\tau & 1 \end{bmatrix}$$

**Solution to 5.** The solution is similar to the problem 4.

$$\begin{aligned} x' &= p(t)x + g(t) \qquad (1)\\ \mathbb{P}(t,t_0) &= \int_{t_0}^{t} p(s)ds \\ x(t) &= \exp\left\{\mathbb{P}(t,t_0)\right\} x_0 + \int_{t_0}^{t} \exp\left\{\mathbb{P}(t,s)\right\} g(s)ds \\ x(t_0) &= x_0 \end{aligned}$$

$$\begin{cases} x_1' = x_1 + tx_2 \\ x_2' = 2x_2 \end{cases} . \; x' = Ax \, , \; A = \begin{bmatrix} 1 & t \\ 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \Phi(t,\tau) &= (\pi_1(t,\tau), \pi_2(t,\tau)). \\ \frac{\partial}{\partial t}\pi_1 &= A\pi_1; \qquad \frac{\partial}{\partial t}\pi_2 = A\pi_2 \end{aligned}$$

$\pi_1(\tau,\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\pi_2(\tau,\tau) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We solve first the equation for $x_2(t)$ with initial data $x_2(\tau)$ :

$$x_2(t) = x_2(\tau)\exp(2(t-\tau))$$

and then the equation for $x_1(t)$ with initial data $x_1(\tau)$ and substituting the solution for $x_2(t) = x_2(\tau)\exp(2(t-\tau))$into the right hand side of the equation, both accoring to the formula in (1)

$$\begin{aligned} x_1(t) &= x_1(\tau)\exp(t-\tau) + \int_{\tau}^{t} \exp(t-s)\left[s\, x_2(\tau)\exp(2(s-\tau))\right] ds \\ &= x_1(\tau)\exp(t-\tau) + x_2(\tau)\exp(t-2\tau)\int_{\tau}^{t}\exp(s)sds = \\ &\left\{\int_{\tau}^{t}\exp(s)sds = te^{t} - \tau e^{\tau} - (e^{t} - e^{\tau})\right\} \\ &= x_1(\tau)\exp(t-\tau) + x_2(\tau)\left(e^{t-\tau} - \tau e^{t-\tau} - e^{2(t-\tau)} + te^{2(t-\tau)}\right) \end{aligned}$$

and substitute particular initial data for $\pi_1(t,\tau), \pi_2(t,\tau)$:

$$\Phi(t,\tau) = \begin{bmatrix} \exp(t-\tau) & \exp(t-\tau)(1-\tau) + \exp(2\,(t-\tau))(t-1) \\ 0 & \exp(2(t-\tau)) \end{bmatrix}$$

**Solution to 6.**

Suppose that every solution of $x' = A(t)x$ is bounded for $t \geq 0$ and let $\Psi(t)$ be a fundamental matrix solution. Prove that $\Psi^{-1}(t)$ is bounded for $t \geq 0$ if and only if the function $t \to \int_0^t \mathrm{tr}A(s)ds$ is bounded from below.

**Hint:** The inverse of a matrix is the adjugate of the matrix divided by its determinant, namely $\Psi^{-1}(t) = [\det(\Psi(t))]^{-1}[Adj(\Psi(t))]$. The adjugate $Adj(B) = \left(\widetilde{B}\right)^T$ where the matrix $\widetilde{B}$ is a matrix of the same size as $B$ with elements in $\widetilde{B}_{ij}$ calculated as $n-1$ dimentional determinants of the matrix $B$ with eliminated $i$-th row and $j$-th column times $(-1)^{i+j}$. See http://en.wikipedia.org/wiki/Adjugate_matrix

The fact that all solutions to the ODE are bounded for $t \geq 0$ implies that all elements in $\Psi(t)$ are bounded for $t \geq 0$ and therefore all elements of $Adj(\Psi(t))$ are bounded for $t \geq 0$ since they consist of sums of products of bounded elements in $\Psi(t)$ times $\pm 1$. It implies that $\Psi^{-1}(t)$ is bounded (has bounded elements) for $t \geq 0$ if and only if $[\det(\Psi(t))]^{-1}$ is bounded that is equivalent to that $|\det(\Psi(t))|$ is bounded from below for for $t \geq 0$. Abel's formula gives that $\det(\Psi(t)) = \det(\Psi(0)) \exp\left(\int_0^t \mathrm{tr}A(s)ds\right)$ and that $|\det(\Psi(t))| = |\det(\Psi(0))| \exp\left(\int_0^t \mathrm{tr}A(s)ds\right) > a > 0$, (bounded from below) if and only if $\ln\left(|\det(\Psi(0))|\right) + \left(\int_0^t \mathrm{tr}A(s)ds\right) > \ln a$ or

$$\left(\int_0^t \mathrm{tr}A(s)ds\right) > \ln a - \ln\left(|\det(\Psi(0))|\right)$$

It implies that $|\det(\Psi(t))|$ is bounded from below if and only if $\int_0^t \mathrm{tr}A(s)ds$ is bounded from below for $t \geq 0$ (cannot go to $-\infty$ with $t_k \to +\infty$ for some for some sequence of times $\{t_k\}_{k=1}^{\infty}$ ).

**Solution to 7**.

**Formulation of the problem.** Suppose that the linear system $x' = A(t)x$ is defined on an open interval $(a, \omega)$ containing the origin whose right-hand end point is $\omega \leq \infty$ and the norm of every solution has a finite limit as $t \to \omega$.

Show that there is a solution converging to zero as $t \to \omega$ if and only if $\int_0^{\omega} \mathrm{tr}A(s)ds = -\infty$.

**Hint**: Use Abels formula and the fact that a matrix has a nontrivial kernel if and only if its determinant is zero.

**Solution.** We show first implication $\Longleftarrow$, that if $\int_0^{\omega} \mathrm{tr}A(s)ds = -\infty$, then there must exist a solution converging to zero with other conditions satisfied.

Suppose opposite, that all solutions $x(t)$ to the system have a limit of the norm strictly positive: $\lim_{t\to\omega} \|x(t)\| \to C_x > 0$ (we remind the condition in the problem that all solutions have a limit $\lim_{t\to\omega} \|x(t)\|$).

There must exist a monotone sequence $\{t_k\}_k^{\infty}$ that converges to $\omega$: $\lim_{k\to\infty} t_k = \omega$, such that $\Phi(t, 0)$ has a limit along this sequence of times: $\lim_{k\to\infty} \Phi(t_k, 0) = \Phi_*$. It follows from the property that any bounded sequence in a complete vector space must have a convergent subsequence and from the fact that columns in $\Phi(t, 0)$ are uniformly bounded solutions to the ODE.

The condition that $\int_0^{\omega} \mathrm{tr}A(s)ds = -\infty$ means that $\lim_{t\to\omega} \int_0^t \mathrm{tr}A(s)ds = -\infty$. The Abel-Liouville formula implies that

$$\lim_{t\to\omega} \det(\Phi(t,0) = \det(\Phi(0,0)) \exp\left(\lim_{t\to\omega} \int_0^t \mathrm{tr}A(s)ds)\right) = 0$$

Therefore for the sequence $\{t_k\}_k^{\infty}$ it follows that

$$0 = \lim_{k\to\infty} \det \Phi(t_k, 0) = \det \lim_{k\to\infty} \Phi(t_k, 0) = \det \Phi_* = 0$$

We conclude that the limit matrix $\Phi_*$ has a non-trivial kernel, namely there is a vector $c = [c_1, c_2, ..., c_n]^T$ such that $\Phi_* c = 0$.

Therefore $\lim_{k\to\infty} \Phi(t_k, 0)c = \Phi_* c = 0$. It means that the solution $x_*(t)$ to the system $x' = A(t)x$ with initial condition $x_*(0) = c$ has the property $\lim_{k\to\infty} x_*(t_k) = 0$ (here $\lim_{k\to\infty} t_k = \omega$)

It contradicts to our supposition that all solutions have $\lim_{t\to\omega} \|x(t)\| \to C_x > 0$.■

The implication $\Longrightarrow$ in the exercise means that if there is a solution $x_*(t)$ such that $\lim_{t\to\omega} x_*(t) = 0$ then $\int_0^{\omega} \mathrm{tr}A(s)ds = -\infty$.

Consider $\xi = x_*(0)$. Consider a basis $\{b_i\}_{i=1}^n$ in $\mathbb{R}^n$ with $b_1 = \xi$. Consider the matrix $W(t)$ (Wronskian) that has columns that are solutions to the system $x' = A(t)x$ with initial conditions $\{b_i\}_{i=1}^n$. Then the Abel formula for $W(t)$ reads:

$$\det(W(t)) = \det\left(W(0)\right) \exp\left(\int_0^t \mathrm{tr}A(s)ds)\right)$$

The limit of this determinant with $t$ tending to $\omega$ is zero

$$\lim_{t\to\omega}\det(W(t))=0$$

because the first column $x_*(t)$ in the Wronsky matrix $W(t)$ tends to zero. It is possible if and only if $\lim_{t\to\omega}\int_0^t \mathrm{tr}A(s)ds=\int_0^\omega \mathrm{tr}A(s)ds=-\infty$ because

$$0=\lim_{t\to\omega}\det(W(t)=\det(W(0))\exp\left(\lim_{t\to\omega}\int_0^t \mathrm{tr}A(s)ds)\right)$$

and columns in $W(0)$ are linearly independent and therefore $\det(W(0))\neq 0.$■

**Solution to 7a.(Corollary 2.33, p. 59)**

Show that if $\liminf_{t\to+\infty}\int_{t_0}^t tr(A(s))ds=+\infty$ then the equation $x'=A(t)x$ has an unbounded solution. Hint: use Abel's formula.

**Solution.**We remind that the transfer matrix $\Phi(t,\tau)$ satifies the initial value problem:

$$\begin{aligned}\frac{d}{dt}\Phi(t,\tau) &= A(t)\Phi(t,\tau)\\ \Phi(\tau,\tau) &= I\end{aligned}$$

Arbitrary solution to the initial problem $x'(t)=A(t)x(t)$, $x(\tau)=\xi$ will be expressed as

$$x(t)=\Phi(t,\tau)\xi$$

According to Abel - Liouville's formulaand Euler formula for complex numbers

$$\begin{aligned}|\det(\Phi(t,0))| &= \left|\det(\Phi(0,0))\exp\left(\int_0^t tr(A(s)ds\right)\right| =\\ \left|\exp\left(\int_0^t tr(A(s)ds\right)\right| &= \left|\exp\left(\mathrm{Re}\left(\int_0^t tr(A(s)ds\right)\right)\right|\end{aligned}$$

Therefore, if $\mathrm{Re}\left(\int_0^p tr(A(s)ds\right)>0$ then

$$|\det(\Phi(p,0))|=\left|\exp\left(\mathrm{Re}\int_0^p tr(A(s)ds\right)\right|>1.$$

On the other hand $\det(\Phi(p,0))$ is a product of eigenvalues $\mu_k$ to the monodromy matrix $\Phi(p,0)$ with multiplicities $m_k$ (it follows from the structure of similar Jordan matrix)

$$|\det(\Phi(p,0)|=\prod_{k=1}^{s}|\mu_k|^{m_k}>1$$

To have this product greater than 1 we must have at least one eigenvalue $\mu_p$ with $|\mu_p|>1$. Therefore, according to one of Floquet theorems, there is a solution $x(t)$ that is not bounded and therefore $\limsup_{t\to\infty}\|x(t)\|=\infty.$ ■

**Solution to 9.**

Abel's formula for fundamental matrix solution is $\det(\Psi(t))=\det(\Psi(0))\exp\left(\int_0^t \mathrm{tr}A(s)ds\right)$. For

$\det(\exp(tA))=\det(I)\exp\left(\int_0^t \mathrm{tr}Ads\right)=\exp\left(t\,\mathrm{tr}A\right)$

$\det((I+\varepsilon A)+O(\varepsilon^2))=\det((I+\varepsilon A)+O(\varepsilon^2)-\exp(\varepsilon A)+\exp(\varepsilon A))=\det(\exp(\varepsilon A)+O(\varepsilon^2))=\det\left(\exp\left(\varepsilon\,\mathrm{tr}A\right)\right)+O(\varepsilon^2)$

$= \exp(\varepsilon\, \mathrm{tr} A) + O(\varepsilon^2) = 1 + \varepsilon\, \mathrm{tr}\,(A) + O(\varepsilon^2)$.

One can also give a direct proof considering an expansion of $\det((I + \varepsilon A) + O(\varepsilon^2))$ according to the addition rool for determinants and observing that the only terms of order zero and one in $\varepsilon \to 0$ in the determinant are 1 and $\varepsilon\, A_{ii}$. Adding the last ones leads to $\varepsilon\, \mathrm{tr}\,(A)$.

**Solution to 10.**

Consider the flow $\phi(t,x)$ corresponding to the autonomous equation $y' = f(y)$, $y \in \mathbb{R}^n$ mapping the domain $\Omega$ to the domain as $\Omega_t = \{y = \phi(t,x), x \in \Omega\}$ where $y$ is the solution to the ODE $y' = f(y)$ with initial data $y(0) = x \in \Omega$.

Show that $\frac{d}{dt}(Vol(\Omega_t)) = \int_{\Omega_t} \mathrm{div}\,(f)\, dy$. **Hint:** use the result of Ex.9.

$(Vol(\Omega_t)) = \int_{\Omega_t} dx$

Considering derivative of the integral is useful to have the integration over a fixed domain and function under the integral depending on time. To implement this idea we introduce a change of variables such that the domain of integration for time $t$ coinsides with the "initial" domain $\Omega_0$ and

$(Vol(\Omega_t)) = \int_{\Omega_t} dx = \int_{\Omega_0} \left|\det\left[\frac{D\phi(t,x)}{Dx}\right]\right| dx$

Consider this integral for $t \to 0$.

$\frac{D}{Dx}\phi(t,x) = \frac{D}{Dx}\left[I\,x + t\,f(0,x) + O(t^2)\right] = \left[I + t\frac{D}{Dx}\,f(0,x) + O(t^2)\right]$, for $t \to 0$

$\det\left[\frac{D}{Dx}\phi(t,x)\right] = \det\left[I + t\frac{D}{Dx}\,f(0,x) + O(t^2)\right] = 1 + t\,\mathrm{tr}\left[\frac{D}{Dx}\,f(0,x)\right] + O(t^2) \geq 0$, for $t \to 0$

and $\left|\det\left[\frac{D\phi(t,x)}{Dx}\right]\right| = \det\left[\frac{D}{Dx}\phi(t,x)\right]$

$\mathrm{tr}\left[\frac{D}{Dx}\,f(0,x)\right] = div\,(f(0,x))$

$\frac{d}{dt}\left(Vol(\Omega_t)\right)_{t=0} = \int_{\Omega_0} div\,(f(0,x))\,dx$

The same argument works naturally for any time $t$.

**Solution to 11.**

11. Show directly that the area of a unit disk is preserved when it is transformed forward to 2 time units by the flow, corresponding to the system $x' = y$, $y' = x$. **Hint**: consider the system in new variables $x + y$ and $x - y$.

**Solution.**

Remind first that the ormula for transformation of area and volume integrals under a transformation of variables.

Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ mapping (havng continuous derivatives) and $\mathcal{A} \subset \mathbb{R}^n$, $\mathcal{B} \subset \mathbb{R}^n$, $\mathcal{B} = \phi(A)$.Then

$$\int_{\phi(\mathcal{A})} f(z)dz = \int_{\mathcal{A}} f(x)\det(J(\phi(x))dx$$

where $J(\phi(x))$ is the Jacoby matrix of the mapping $\phi$.

For volume of the transformed set $\phi(A)$ the formula simplifies by inserting $f(z) = 1$:

$$Vol(\phi(\mathcal{A})) = \int_{\phi(\mathcal{A})} dz = \int_{\mathcal{A}} |\det(J(\phi(x))|\,dx$$

For a linear transformation of $\mathcal{A}$ generated by the linear equation with the transfer matrix $\Phi(t,0)$ the Jacoby matrix for each time $t$ is the transfer matrix itself:

$$J(\phi(x)) = \Phi(t,0)$$

and this formula for the volume of the transformed set $\mathcal{A}$ looks as:

$$\begin{aligned} Vol(\Phi(t,0)(\mathcal{A})) &= \int_{\Phi(t)(\mathcal{A})} dz = \int_A |\det(\Phi(t,0))|\,dx \\ & \int_A \left|\det(\overbrace{\Phi(0,0)}^{=I})\exp\left(\int_0^t tr(A(s)ds\right)\right| dx \\ &= \int_A \left|\exp\left(\int_0^t \mathrm{tr}(A(s)ds\right)\right| dx \end{aligned}$$

The particular problem here is reduced to calculating the determinant of the transfer matrix of the given system of equations:

$$
\begin{aligned}
x' &= y, \\
y' &= x.
\end{aligned}
$$

or

$$
\begin{aligned}
r' &= Ar \\
A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
\mathrm{tr}(A) &= 0
\end{aligned}
$$

with constant matrix $A$, that simplifies the formula for the volume even more.

$$
\begin{aligned}
Vol(\Phi(t,0)(\mathcal{A})) &= \int_A \left| \exp\left( \int_0^t [\mathrm{tr} A]\, ds \right) \right| dx = \\
&= \int_A \left| \exp\left( t\, [\mathrm{tr} A]\, ds \right) \right| dx = \int_A \left| \exp\left( t\, [\mathrm{tr} A]\, ds \right) \right| dx \\
&= \int_A dx = Vol(\mathcal{A})
\end{aligned}
$$

It shows that the transfer mapping of this system (or flow corresponding to this system as one says in the theory of dynamical systems) preserves volume. It implies that any disc will preserve it's arean under this mapping in fact at any time.