May 2, 2022

Lecture 12
Limit sets (attractors), positively invariant sets, periodic solutions, limit cycles for non-linear autonomous ODEs - dynamical systems.

Plan

- Semi - orbits. Limit sets. p. 142. Positively (negatively) invariant sets p. 142.
- Existence of an equilibrium point in a compact positively invariant set. Theorem 4.45, p. 150.
- Planar systems. Periodic orbits. Poincare-Bendixson theorem. (only idea of the proof is discussed) Theorem 4.46, p. 151.
- Examples on applications of Poincare-Bendixson theorem.
- Generalized Poincare-Bendixson theorem. (missed in the course book, only formulation is given)

0.1 Introduction to limit sets and their properties.

We consider flows or dynamical systems corresponding to autonomous differential equations

$$\dot{x} = f(x), \quad f : G \to \mathbb{R}^N, \quad x(0) = \xi \tag{1}$$

with f locally Lipschitz and denote by $\varphi(t, \xi)$ the transition mapping or the local flow generated by f. For $\xi \in G$ let $I_\xi = (\alpha_\xi, \omega_\xi)$ denote the open maximal interval - the interval of existence of maximal solution to (1).

Definition. (Positive semi-orbit)

We denote by $O(\xi)$ the orbit of the solution to (1), $O(\xi) = \{x(t) : t \in (\alpha_\xi, \omega_\xi)\}$.

We define the positive semi-orbit $O_+(\xi) = \{x(t) : t \in [0, \omega_\xi)\}$ of ξ - for future, and negative semi-orbit (for the past) $O_-(\xi) = \{x(t) : t \in (\alpha_\xi, 0]\}$ o$\not{f}$ ξ - for the past.

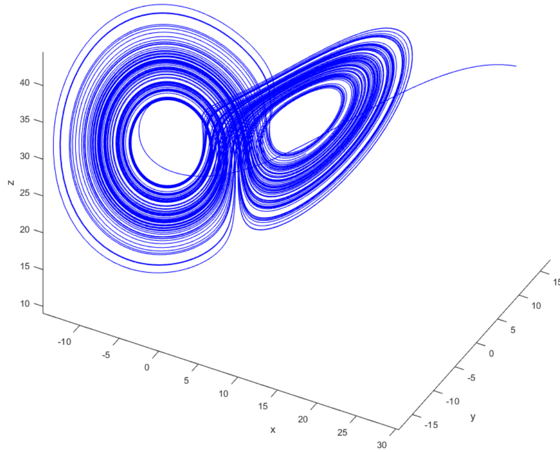
Definition. (Limit point of ξ)

• A point $z \in R^N$ is called an ω - limit point of ξ (or of it's positive semi-orbit $O_+(\xi)$ or it's orbit $O(\xi)$), if there is a sequence of times $\{t_n\} \in [0, \omega_\xi)$ tending to the "maximal time in the future", $t_n \nearrow \omega_\xi$ such that $\varphi(t_n, \xi) \to z$ as $n \to \infty$

• Similarly a point $z \in R^N$ is called an α - limit point of ξ (or it's negative semi-orbit $O_-(\xi)$ or it's orbit $O(\xi)$) if there is a sequence of times $\{t_n\} \in (\alpha_\xi, 0]$ tending to the "minimal time in the past", $t_n \searrow \alpha_\xi$ such that $\varphi(t_n, \xi) \to z$ as $n \to \infty$.

Definition. (ω **- limit set)**

The ω - limit set $\Omega(\xi)$ of ξ (or it's positive semi-orbit $O_+(\xi)$ or it's orbit $O(\xi)$) is the set of all it's ω- limit points (in future) of ξ.



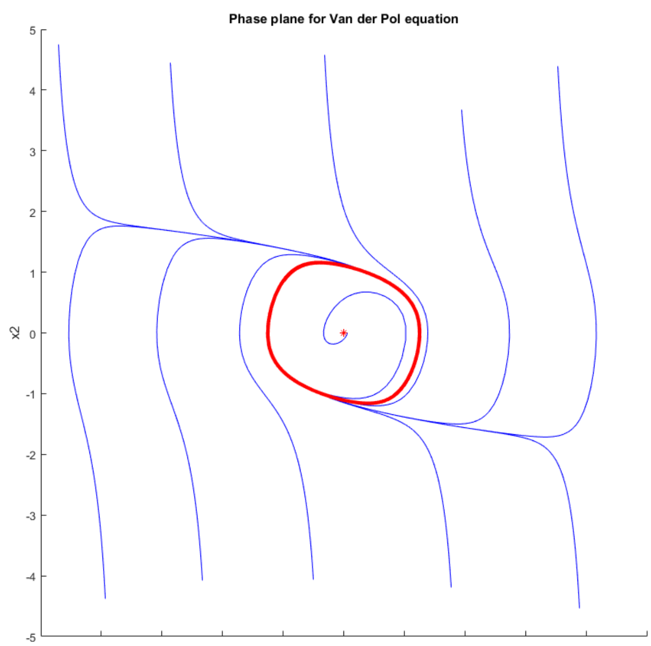
A trajectory approaching the ω - limit set of the Lorentz system

$$
\begin{aligned}
x' &= -\sigma(x-y) \\
y' &= rx - y - xz \\
z' &= xy - bz
\end{aligned}
$$

for $\sigma = 10$, $r = 28$, $b = 8/7$.

Definition

The α - limit set $\Omega(\xi)$ of ξ (or it's negative semi-orbit $O_{-}(\xi)$ or it's orbit $O(\xi)$) is the set of all it's α- limit points (in the past).



ω -limit set that is a periodic orbit.

Definition. Positively invariant set (it is often called ω - invariant set **)**

A set $U \subset G$ is said to be positively invariant under the local flow $\varphi(t, \xi)$ generated by f if for each starting point $\xi \in U$ from U the corresponding positive semi - orbit $O_{+}(\xi)$ is contained in U.

It means that all trajectories $x(t)$ starting in U stay in U as long as they exist in future.

One defines sets negatively invariant similarly, but with respect to the past.

Positively invariant sets are sometimes rather naturally called ω - invariant sets.

Definition

One also says that the set U is just invariant with respect to the flow $\varphi(t, \xi)$ if $O(\xi) \subset U$ for all $\xi \in U$. It means that all trajectories going through ξ belong to U both in the "whole past" and in the "whole future".

Remark

We know that compact positively invariant sets include trajectories that have "infinite" maximal existence time in the future: $J \cap [0, \infty)$. It makes it meaningfull to investigate limit sets of trajectories that are contained especially in compact positively invariant sets.

The first step in this kind of investigation is to identify possibly small positively invariant sets, that localize solutions. The second step is to classify and to identify ω - limit sets that can be contained there. It particular one is interested in fining ω - limit sets for particular given systems.

0.2 Methods of hunting positively - invariant sets (there is a separate pdf file with this text)

A system of ODEs has naturally many positively - invariant sets, for example the whole domain G is always a positively - invariant set, but it is not very interesting. We like to find possibly narrow positively invariant sets showing more precisely where trajectories or solutions to the equation tend when time t tends to the upper bound of the maximal time interval.

How to find a positively - invariant set?

Method 1. A general idea that is used to answer many questions about behaviour of solutions (trajectories) to ODEs, is the **idea of test functions**. One checks if the velocities $f(x)$ are directed inside or outside with respect to the sets like $Q = \{x \in U : V(x) \leq C\}$ or $Q = \{x \in U : V(x) \geq C\}$ defined by some simple test functions $V : U \to \mathbb{R}$, $U \subset G$. The advantage of the idea with test functions is that one does not need to solve the equation to use it.

• It is convenient to find a test function $V(x)$ that has a level set $\partial Q = \{x : V(x) = C\}$ that is a closed curve (or surface in higher dimensions) enclosing a bounded domain Q.

Typical examples are $V(x,y) = x^2 + y^2 = R^2$ - circle or radius R, or $V(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ - ellipse,

or more complicated ones as $V(x,y) = x^6 + ay^4$ - smoothed rectangle shape or squeezed ellipse.

'The equation $V(x,y) = x^2 + xy + y^2 = C$ - gives an ellipse rotated in $\pi/4$ and having axes A and B related as $A/B = \sqrt{3}$etc.

• To show that a particular level set ∂Q bounds an positively - invariant set Q we check the sign of the directional derivative V_f of V along the velocity $f(x)$ in the equation $x' = f(x)$:

$$dV(x(t))/dt = V_f(x) = (\nabla V \cdot f)(x)$$

for all points on the level set $\{V(x) = C\}$ for a particular constant C.

• Point out that the gradient $\nabla V(x)$ is the normal vector to the level set $\{V(x) = C\}$ that goes through the point x. Therefore the sign of $V_f(x)$ shows if the trajectories towards the same side of the level set as the gradient ∇V (if $V_f(x) > 0$) or towards the opposite side (if $V_f(x) < 0$).

• Then if $V(x)$ is rising for x going out of Q, and $V_f(x) < 0$ then the domain Q inside this level set ∂Q (curve in the plane case) will be positively - invariant. Similarly if $V(x)$ is decreasing out of this level set, and $V_f(x) < 0$ on the level set ∂Q then the domain Q inside this level set will be positively - invariant.

In the opposite case the complement to Q that is $\mathbb{R}^N \backslash Q$ will be positively - invariant and trajectories $\varphi(t,\xi)$ starting in this complement: $\xi \in \mathbb{R}^N \backslash Q$ will never enter Q.

First integrals. A very particular case of test functions are functions that are constant on all trajectories $\varphi(t,\xi)$ of the system. It means that $\frac{d}{dt} V(\varphi(t,\xi)) = V_f(x) = (\nabla V \cdot f)(x) \equiv 0$. In this case all level sets of the first integral are invariant sets, because velocities $f(x)$ are tangent vectors to the level sets in this case. Such functions are called **first integrals** and represent conservation laws in ODEs. Usually but not always, such test functions have the meaning of the total energy in the system. In this particular case each level set is a union of orbits to the differential equation.

An interesting property of first integrals is that their level sets consist of orbits.

Method 2. If it is sometimes difficult to guess a simple test function giving one closed formula for the boundary of an positively - invariant set as in the Method 1, then one can try to identify a boundary for a positively - invariant set as a curve (or a surface in higher dimensions) consisting of a number of simple peaces, for example straight segments.

The simplest positively - invariant set of such kind would be a rectangle (a rectangular box in higher dimensions) with sides parallel to coordinate axes. Then a simple check that this rectangle is a positively - invariant is just to check the sign of x_1 or x_2 - components of $f(x)$ on these segments, showing that trajectories go inside or outside of the rectangle.

A bit more complicated analysis is necessary to show that no trajectories can approach these segments in finite time (if one of the segments belongs to the boundary ∂G of G where the equation is not defined).

Application to Poincare Bendixson theorem

One searches often positively - invariant sets with special properties. For example to apply the Poincare-Bendixson theorem for systems in the plane formulated later in this course, one needs to find a positively - invariant set that does not contain any equilibrium points.

Example of finding positively - invariant sets and ω - limit sets with help of polar coordinates and the simple test function $r(x,y) = \sqrt{x^2+y^2}$.

Consider the system

$$\begin{aligned} x' &= -ay + f(r)x \\ y' &= ax + f(r)y \end{aligned}$$

where $r = \sqrt{x^2+y^2}$. We will try to find an explicit expression for the corresponding flow by introducing polar coordinates $x = \cos(\theta)r$, $y = \sin(\theta)r$. We differentiate $r(t)$ using expressions for r and for x', y' in the equation, and arrive to following formulas:

$$\begin{aligned} \left(r(t)^2\right)' &= 2rr' = (x^2+y^2)' = 2xx' + 2yy' \\ &= 2x(-ay + f(r)x) + 2y\,(ax + f(r)y) = 2f(r)\left(x^2+y^2\right) = 2f(r)r^2 \end{aligned}$$

Therefore:

$$r' = f(r)r$$

The equation for the polar angle θ can be derived by differentiating $\tan(\theta(t))$:

$$\begin{aligned} (\tan(\theta(t)))' &= \theta'\left(\frac{1}{\cos^2(\theta)}\right) = \left(\frac{y}{x}\right)' = \frac{y'x - x'y}{x^2} \\ &= \frac{ax^2 + f(r)xy - (-ay^2 + f(r)xy)}{x^2} = \frac{ax^2 + ay^2}{x^2} = \frac{a(x^2+y^2)}{x^2} = \frac{a}{\cos^2\theta} \end{aligned}$$

Therefore

$$\theta' = a$$

The equation for $r(t)$ can be solved by integration.

Each positive root r_* to $f(r)$ corresponds to a periodic trajectoty $r(t) = const = r(0) = r_*$, $\theta(t) = \theta(0) + at$

This periodic orbit will attract trajectories, that start nearby if $\frac{df}{dr}(r_*) < 0$. For $r < r_*$ inside the circle $r = r_*$ $f(r) > 0$ and correspondingly $r' = f(r)r > 0$. For $r > r_*$ outside the circle $r = r_*$ we have $f(r) < 0$ and correspondingly $r' = f(r)r < 0$. It implies that the circle

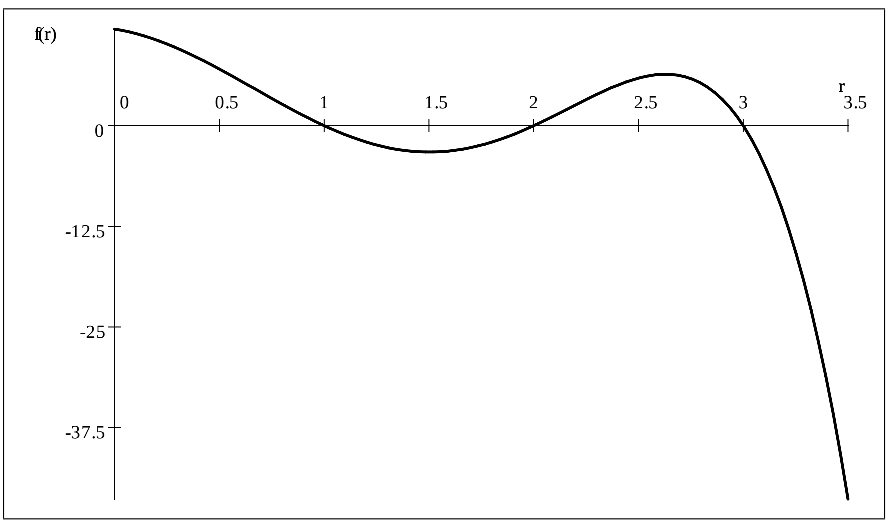
$r = r_*$ will be an ω - limit set $\Omega(\xi)$ for points ξ close to the circle $r = r_*$).

If r_* is a root of f where the first term in Taylor series is $c(r - r_*)^2$ with $c > 0$, then nearby trajectories will be attracted to the periodic orbit $r(t) = r_*$ from inside, and will run away from the periodic orbit from the outside of it.■

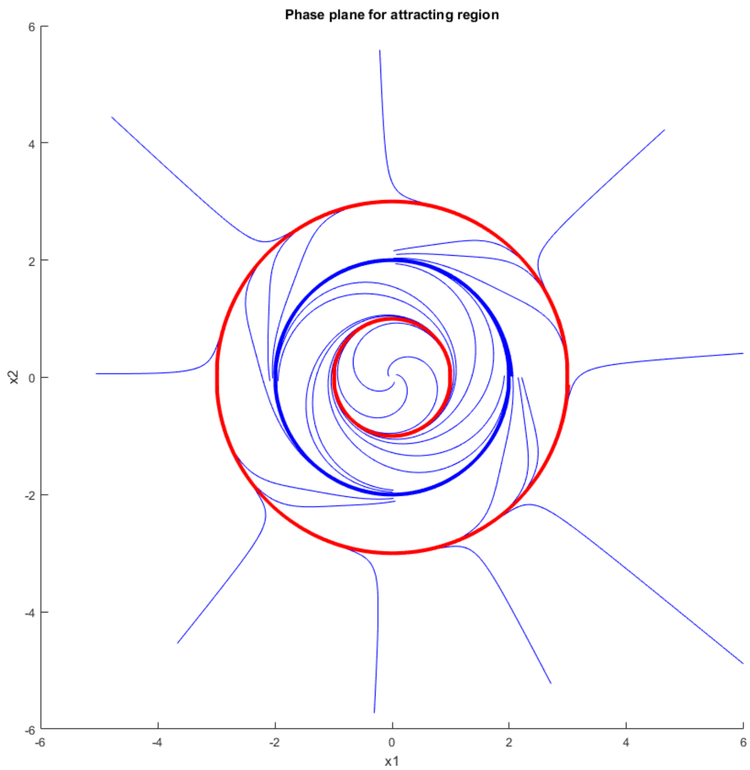
Example.

An example of this type with tree periodic solutions, orbit $r = 1$ and $r = 3$ (red) are ω - limit sets for points ξ close to the set $r = 1$ and to the set $r = 3$, the orbit of one of them with $r = 2$ (blue) is an α - limit set for points ξ close to the set where attractor consists of limit points $r = 2$:

$$\begin{aligned} f(r) &= (1 - r^2)(3 - r)(4 - r^2) \\ a &= -10 \end{aligned}$$



We have $dr/dt = f(r)r$


Phase plane for attracting region
x1
x2
6
4
2
0
-2
-4
-6
-6
-4
-2
0
2
4
6

In the following example from the course book such kind of system is considered for one more particular function $f(r)$.

Exercise 4.16, p. 140.

Exercise 4.16

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(z) = f(z_1, z_2) := \big(z_2 + z_1(1 - \|z\|^2)\,,\ -z_1 + z_2(1 - \|z\|^2)\big).$$

Show that f generates a local flow $\varphi\colon D \to \mathbb{R}^2$ given by

$$\varphi(t,\xi) = \big(\|\xi\|^2 + (1 - \|\xi\|^2)e^{-2t}\big)^{-1/2} R(t)\xi\,,$$

where the function $R\colon \mathbb{R} \to \mathbb{R}^{2\times 2}$ is given by

$$R(t) := \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad \forall\, t \in \mathbb{R}. \tag{4.27}$$

(and so $R(t)\xi$ is a clockwise rotation of ξ through t radians) and

$$D := \big\{(t,\xi) \in \mathbb{R} \times \mathbb{R}^2 \colon \|\xi\|^2 + (1 - \|\xi\|^2)e^{-2t} > 0\big\}.$$

(*Hint*. Show that, for $\xi = (\xi_1, \xi_2) \neq 0$, the initial-value problem (4.25) may be expressed – in polar coordinates – as

$$\dot r(t) = r(t)\big(1 - r^2(t)\big), \quad \dot\theta(t) = -1, \quad (r(0), \theta(0)) = (r^0, \theta^0),$$

where $r^0 = \|\xi\|$, $r^0 \cos\theta^0 = \xi_1$ and $r^0 \sin\theta^0 = \xi_2$.)

Solution. The equations in polar form follow from the general argument above.

We solve the equation for r :

$$\begin{aligned} \frac{dr}{dt} &= r(1 - r^2) \\ \frac{dr}{r(1 - r^2)} &= dt \end{aligned}$$

$$\frac{1}{r(1 - r^2)} = \frac{1}{r} - \frac{1}{2\,(r + 1)} - \frac{1}{2\,(r - 1)}$$

$$\begin{aligned}
\int \frac{dr}{r(1-r^2)} &= \ln r - \frac{1}{2}\ln\left(r^2-1\right) \\
\ln r - \frac{1}{2}\ln\left(r^2-1\right) &= t + C \\
C &= \ln|\xi| - \frac{1}{2}\ln\left(|\xi|^2-1\right) \\
\ln r - \frac{1}{2}\ln\left(r^2-1\right) - \left(\ln|\xi| - \frac{1}{2}\ln\left(|\xi|^2-1\right)\right) &= t
\end{aligned}$$

$$\begin{aligned}
\exp(t) &= \exp\left(\ln r - \frac{1}{2}\ln\left(r^2-1\right) - \ln|\xi| + \frac{1}{2}\ln\left(|\xi|^2-1\right)\right) \\
\frac{r}{\sqrt{r^2-1}}\frac{\sqrt{|\xi|^2-1}}{|\xi|} &= \exp(t) \\
\frac{(r^2-1)}{r^2}\frac{|\xi|^2}{\left(|\xi|^2-1\right)} &= \exp(-2t) \\
\left(r^2-1\right)|\xi|^2 &= r^2\left(|\xi|^2-1\right)\exp(-2t) \\
r^2(|\xi|^2 + \left(1-|\xi|^2\right)\exp(-2t)) &= |\xi|^2 \\
r^2 &= \frac{|\xi|^2}{(|\xi|^2 + \left(1-|\xi|^2\right)\exp(-2t))} \\
r &= \frac{|\xi|}{\sqrt{(|\xi|^2 - 1 - |\xi|^2\exp(-2t))}}
\end{aligned}$$

Example 4.37. p. 142. Do it as exercise.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be as in the Exercise 4.16, the generator of a local flow considered above.

Let Δ be an open unit disc in $\mathbb{R}^2$, namely $\Delta = \left\{(z_1, z_2) \in \mathbb{R}^2 : z_1^2 + z_2^2 < 1\right\}$.

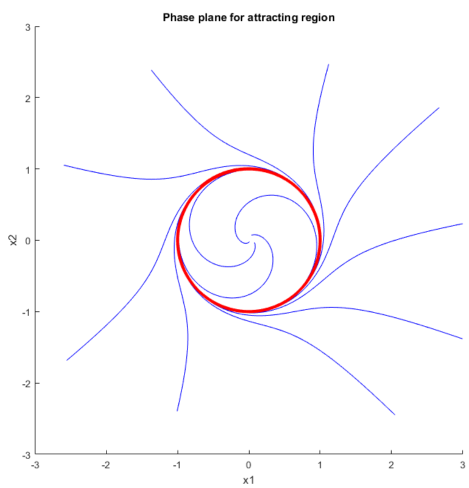
Show that sets Δ, $\partial\Delta, \mathbb{R}^2\backslash\overline{\Delta}$ are invariant and find for every $\xi \in \mathbb{R}^2$ the corresponding ω and α limit set.

Remark. In the case $\|\xi\| > 1$ solutions $\varphi(t,\xi)$ have the maximal interval I_ξ that is not the whole $\mathbb{R}$, but is bounded in the past $I_\xi = (\alpha_\xi, \infty)$.

The calculation of α_ξ using the explicit solution found in the exercise 4.16 is given here:

$$
\begin{aligned}
\|\xi\|^2 + \left(1 - \|\xi\|^2\right) e^{-2t} &= 0 \\
\frac{\|\xi\|^2}{\|\xi\|^2 - 1} &= e^{-2t} \\
\ln\left(\frac{\sqrt{\left(\|\xi\|^2 - 1\right)}}{\|\xi\|}\right) &= t = \alpha_\xi < 0
\end{aligned}
$$

The phase portrait is the following:



0.3 Dynamical systems in plane. Poincare Bendixson theorem, periodic solutions and more positively invariant sets.

Theorem. Poincare-Bendixson theorem.

Suppose that $\xi \in G \subset \mathbb{R}^2$ is such that the closure of the positive orbit $O_+(\xi)$ is compact and is contained in G and the ω - limit set $\Omega(\xi)$ does not contain equilibrium points.

Then the ω - limit set $\Omega(\xi)$ is an orbit of a periodic solution.$\square$

Counterexample: an annulus containitng no periodic orbits, because it is a region of attraction containing an attracting equilibrium.

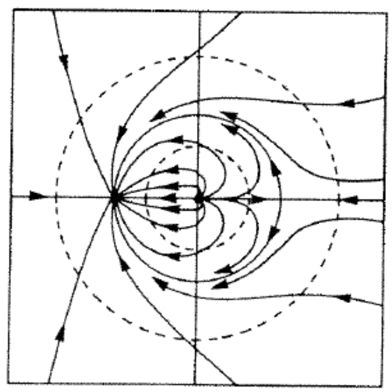


Fig. 3.25. Phase portrait for the system $\dot{r} = r(1-r)$, $\dot{\theta} = \sin\theta$.

Definition of ω - limit cycle

A periodic orbit γ (an orbit corresponding to a periodic solution) is called an ω **- limit cycle** (or often just a limit cycle) if $\gamma = \Omega(\xi)$ for some starting point $\xi \in G\backslash\gamma$: namely that γ is an ω-limit set for some point ξ outside γ.

This definition excludes the case of phase portraits completely filled periodic orbits, as the system

$$
\begin{aligned}
x' &= -y, \\
y' &= x,
\end{aligned}
$$

having all orbits being circles around the origin corresponding to periodic solutions.

Hint to applications. It is difficult to check conditions in the Poincare-Bendixson theorem as they are.

It is much easier to check that there is a compact positively invariant set $C \subset G \subset \mathbb{R}^2$ such that $\xi \in C$. Then the ω - limit set $\Omega(\xi) \subset C$ is not empty. If C contains no equilibrium points, then the closure of $\Omega(\xi)$ cannot contain equilibrium points either and by the Poincare-Bendixson theorem $\Omega(\xi)$ is an orbit of a periodic solution.

One fundamental fact about positively invariant sets is the following.

Theorem 4.45. p. 150, L&R (slightly generalised, without proof)

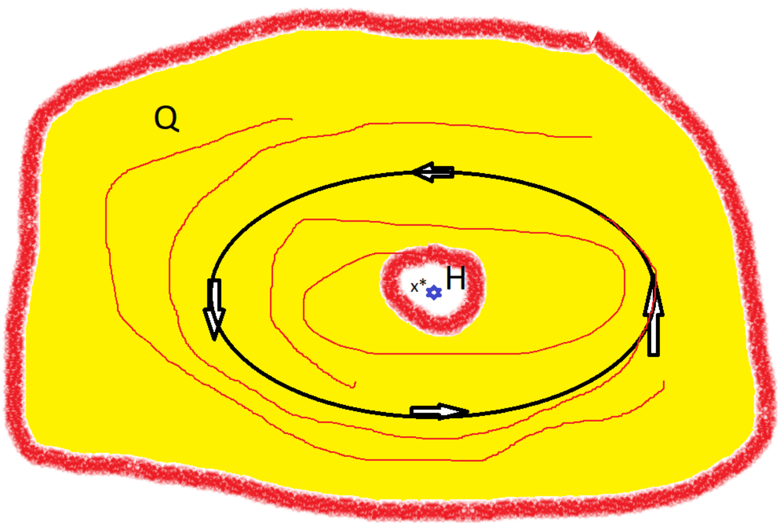
Suppose that $C \subset G \subset \mathbb{R}^2$ is non-empty and compact and is homeomorphic to a circular disc (has no holes). If C is positively invariant under the flow $\varphi(t, \xi)$, then C must contain at least one equilibrium point for the corresponding ODE.

Proof of this theorem is based in the Bohl-Brouwer fixed-point theorem about the existence of fixed points $x = F(x)$ of a continuous mapping $F : C \to C$ for a compact $C \subset \mathbb{R}^n$ homeomorphic to a ball. See an Appendix in L.R.

Definition. Two sets A and B in $\mathbb{R}^n$ are homeomorphic if there is a continuous invertible mapping (homeomorphism) $\Theta : A \to B$,and $\Theta^{-1} : B \to A$.

The Theorem 4.45 has an important practical consequence for the application of the Poincare Bendixson theorem.

Remark. Considering any periodic orbit in the plane $\mathbb{R}^2$ we see that it encloses a compact positively invariant set Q homeomorfic to a round disc (it follows from Jordan's lemma). Theorem 4.45 suggests that Q includes at least one equilibrium point if the differential equation is defined on Q. It means that any periodic orbit in plane must surround at least one equilibrium point. It makes that typical compact positively - invariant set C considered for applying the Poincare-Bendixson theorem should be a closed ring shaped set with at least one hole in the middle including an equilibrium point.



Check list for application of the Poincare-Bendixson theorem for finding periodic solutions.

• One starts with applying one of the two methods above to find a compact positively - invariant set Q.

• Then we consider if Q has an equilibrium inside. Usually there is one such if our intuition is not wrong. Therefore the set Q does not satisfy conditions in the Poincare-Bendixson theorem yet. It is only the first step to the goal.

• Suppose there is just one equilibrium point x_* inside Q. It might be that this equilibrium is asymptotically stable and attracts all trajectories starting in Q. Then there is no periodic orbit inside Q.

• To have a periodic orbit in Q we need to find a "hole" H around the equilibrium x_* such that no trajectories enter it. Then the closure of the set $Q\backslash H$ without the hole will be a compact ring - shaped set (annulus) that is positively invariant and contains no equilibrium points. Then all trajectories $x(t)$ starting in $\overline{Q\backslash H}$ will have a non-empty ω - limit set that according to the Poincary Bendixson theory is a periodic orbit. There can be several periodic orbits in $\overline{Q\backslash H}$ that are ω - limit sets for different trajectories. There can also be some periodic orbits that are not ω - limit sets!

• The "hole"H that repells trajectories can be found using the method of test functions, sometimes using the same test function V as one used to identify Q, just choosing different level sets for Q and for H.

• Alternatively one can use the linearization to show that this equilibrium is a repeller, namely an unstable node or an unstable spiral, and therefore trajectories cannot enter some small neighbourhood of the equilibrium in the middle of the set Q. This method is convenient in the case when the equilibrium is not the origin.

• One must check at the end that the positively invariant annulus (the closed ring shaped domain) does not include equilibrium points (no at the boundary either!).

It is often simpler to do it after carrying out estimates for V_f by first checking zeroes of $V_f(x) = 0$ that contain naturally all equilibrium points but is a scalar equation, and then checking zeroes of the system $f(x) = 0$.

Examples on Poincare-Bendixson's theorem

Example. Show that the following system has a periodic solution.

$$\begin{aligned} x' &= y \\ y' &= -x + \left(1 - x^2 - 2y^2\right) y \end{aligned}$$

The only equilibrium is in the origin. We try to find a compact positively invariant set using the method with test function.

We try the simples test function $V(x,y) = (x^2 + y^2)/2$.

$$\begin{aligned} V_f(x,y) &= \frac{d}{dt} V(x(t), y(t)) = \nabla V(x,y) \cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \\ &= \nabla V(x,y) \cdot f(x,y) = -\left(x^2 + 2y^2 - 1\right) y^2 \le (\ge) 0 \end{aligned}$$

We observe from the expression in the inequality that a particular curve: the ellipse with the equation

$$x^2 + 2y^2 = 1$$

separates points (x,y) where $V_f(x,y) \ge 0$ and $V_f(x,y) \le 0$.

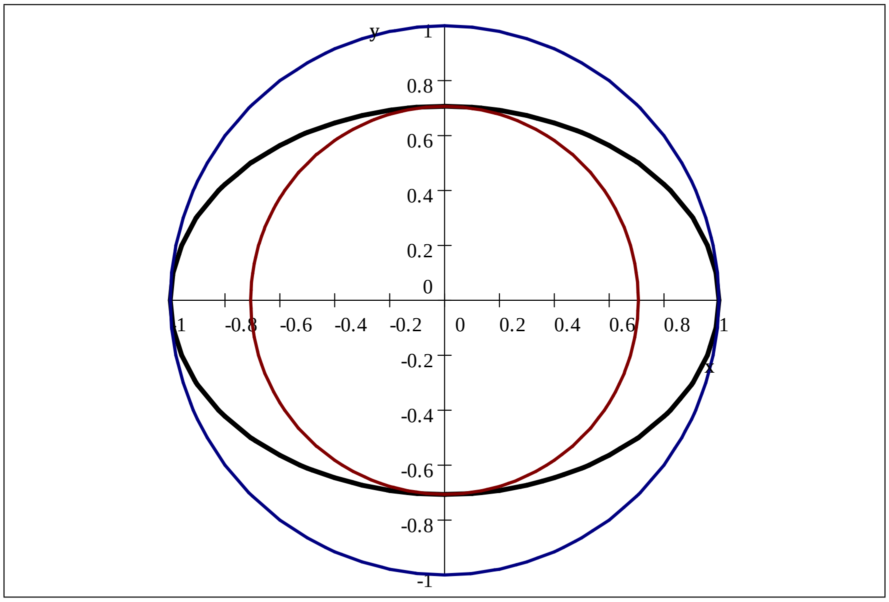
The negative sign of $V_f(x,y)$ says that trajectories go inside the level set of V (a circle in this case) going through the point (x,y).

The positive sign of $V_f(x,y)$ says that trajectories go outside the level set of V going through the point (x,y).

The half axes of ellipse $x^2 + 2y^2 = 1$ are expressed from the transformed equation

$$\frac{x^2}{1^2} + \frac{y^2}{\left(1/\sqrt{2}\right)^2} = 1$$

We find the largest level set (circle) of $V(x,y)$ inside this ellipse (red) $x^2 + y^2 = 1/2$ and the smallest level set of $V(x,y)$ outside this ellipse (blue) $x^2 + y^2 = 1$ to get the smallest positive invariant set $C = \{(x,y) : 1/2 \le x^2 + y^2 \le 1\}$that includes a periodic orbitbecause there are no equilibrium points inside it because the origin is the only equilibrium point for the system.



As Theorem 4.45 and examples considered before suggest, the positively invariant set we look for applying the Poincare Bendixson theorem must have a shape of annulus with a hole in the middle that contains at least one equilibrium point. The next Proposition gives a particular hint how to find the "hole" for such an annulus domain with less effort by using the Grobman-Hartman theorem that we studied earlier.

Lecture 12
Proposition about the existence of a limit cycle. Exercises on hunting periodic solutions and limit cycles. More results about ω - limit sets in the plane. Bendixson's theorem about the non-existence of periodic solutions.

Definition of ω - limit cycle

A periodic orbit γ (an orbit corresponding to a periodic solution) is called an ω **- limit cycle** (or often just a limit cycle) if $\gamma = \Omega(\xi)$ for some starting point $\xi \in G \backslash \gamma$: namely that γ is an ω-limit set for some point ξ outside γ.

This definition excludes the case of phase portraits completely filled periodic orbits, as the system

$$
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$$

having all orbits being circles around the origin corresponding to periodic solutions.

Proposition 4.56. p. 165.

Let $C \subset G$ be a compact set that is positively invariant under the local flow (dynamic system) $\varphi(t, \xi)$ generated by the equation $x'(t) = f(x)$. Assume that an the point x_* is an interior point in C and is the only equilibrium point in C. Assume that f is differentiable in x_*. Let A be the Jacoby matrix of f in x_*: $\frac{Df}{Dx}(x_*) = A$. Let eigenvalues of A have both eigenvalues with positive real parts: $\operatorname{Re} \lambda_{1,2} > 0$.

Then there exists at least one ω - limit cycle in C.

□

Proof is an exercise on application of Grobman-Hartman theorem.

Example 4.57

Consider again the system given in Exercise 4.16, with $G = \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(z) = f(z_1, z_2) := \big(z_2 + z_1(1 - \|z\|^2),\, -z_1 + z_2(1 - \|z\|^2)\big)$. Let C be the closed unit disc $\{z \in \mathbb{R}^2 : \|z\| \le 1\}$. Then

$$\langle z, f(z)\rangle = \|z\|^2\big(1 - \|z\|^2\big) = 0 \quad \forall z \in \partial C,$$

and so solutions starting in C cannot exit C in forwards time. Thus, the compact set C is positively invariant. Moreover, 0 is the unique equilibrium in C

and

$$A = (Df)(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

with spectrum $\sigma(A) = \{1 + i, 1 - i\}$. Therefore, by Proposition 4.56, we may conclude the existence of a limit cycle in C. This, of course, is entirely consistent with Exercise 4.16 and Example 4.37, the conjunction of which shows (by explicit computation of the local flow) that the unit circle $\gamma = \partial C$ is a periodic orbit and coincides with the ω-limit set $\Omega(\xi)$ of every ξ with $0 < \|\xi\| < 1$. $\triangle$

Exercise. Rectangular positively invariant set and application of the Poincare Bendixson theorem.

Consider the following system of ODEs :

$$\begin{cases} x' = & 10 - x - \frac{4xy}{1+x^2} \\ y' = & x\left(1 - \frac{y}{1+x^2}\right) \end{cases}$$

a) show that the point (x_*, y_*) with coordinates $x_* = 2$ and $y_* = 5$ is the only equilibrium point and is a repeller;

b) find a rectangle $[0, a] \times [0, b]$ in the first quadrant $x > 0$, $y > 0$ bounded by coordinate axes and by two lines parallel to them, that is a compact positively invariant set. Explain why the system must have at least one periodic orbit in this rectangle.

1. **Solution.**

a) $x_* = 2$ and $y_* = 5$ is an equilibrium point: $\left(1 - \frac{5}{1+2^2}\right) = 0$; and $10 - 2 - \frac{4 \cdot 2 \cdot 5}{5} = 10 - 2 - 8 = 0$.

The Jacobi matrix is $A = \begin{bmatrix} -4\frac{y}{x^2+1} + 8x^2\frac{y}{(x^2+1)^2} - 1 & -4\frac{x}{x^2+1} \\ -\frac{y}{x^2+1} + 2x^2\frac{y}{(x^2+1)^2} + 1 & -\frac{x}{x^2+1} \end{bmatrix}$. It is calculated as:

$$\nabla\left(10 - x - \frac{4xy}{1+x^2}\right) = \left.\begin{bmatrix} -4\frac{y}{x^2+1} + 8x^2\frac{y}{(x^2+1)^2} - 1 \\ -4\frac{x}{x^2+1} \end{bmatrix}\right|_{x=2,\ y=5}$$

$$= \begin{bmatrix} -4\frac{5}{5} + 8\,(4)\,\frac{5}{25} - 1 \\ -4 * \frac{2}{5} \end{bmatrix} = \begin{bmatrix} -4 + \frac{32}{5} - 1 \\ -\frac{8}{5} \end{bmatrix} = \begin{bmatrix} \frac{7}{5} \\ -\frac{8}{5} \end{bmatrix} \begin{bmatrix} 1.4 \\ -1.6 \end{bmatrix}$$

$$\nabla\left(x\left(1 - \frac{y}{1+x^2}\right)\right) = \left.\begin{bmatrix} -\frac{y}{x^2+1} + 2x^2\frac{y}{(x^2+1)^2} + 1 \\ -\frac{x}{x^2+1} \end{bmatrix}\right|_{x=2,\ y=5} = \begin{bmatrix} -\frac{5}{5} + 2(4)\frac{5}{25} + 1 \\ -\frac{2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -1 + \frac{8}{5} + 1 \\ -\frac{2}{5} \end{bmatrix} = \begin{bmatrix} 1.6 \\ -0.4 \end{bmatrix}$$

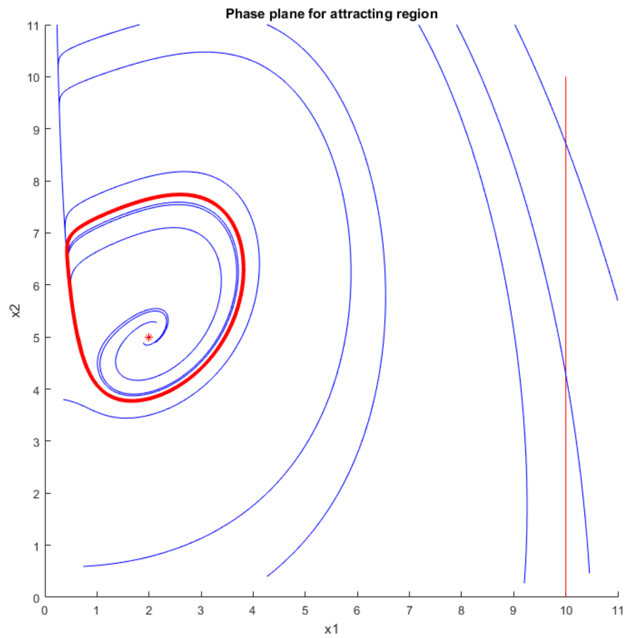
The Jacobi matrix in x_*, y_* is $A = \begin{bmatrix} 1.4 & -1.6 \\ 1.6 & -0.4 \end{bmatrix}$, characteristic polynomial: $\lambda^2 - \lambda + 2 = 0$,

trace$(A) = 1 > 0$, det$(A) = 2 > \frac{[\text{trace}(A)]^2}{4} = \frac{1}{4}$ that corresponds to an unstable spiral and it is a repeller, eigenvalues are: $\lambda_1 = 0.5 + \sqrt{0.25 - 2} = 0.5 + i\sqrt{1.75}$, $\lambda_2 = 0.5 - \sqrt{0.25 - 2} = 0.5 - i\sqrt{1.75}$.

It implies by the Grobman-Hartman theorem, that trajectories cannot enter a small open domain with the center the equilibrium point $(2, 5)$ inside and some small diameter ε. We do not need to (and cannot) specify ε here.

b) Observe that the closed first quadrant is a positively invariant set.

$$\begin{cases} x' = & 10 - x - \frac{4xy}{1+x^2} \\ y' = & x\left(1 - \frac{y}{1+x^2}\right) \end{cases}$$



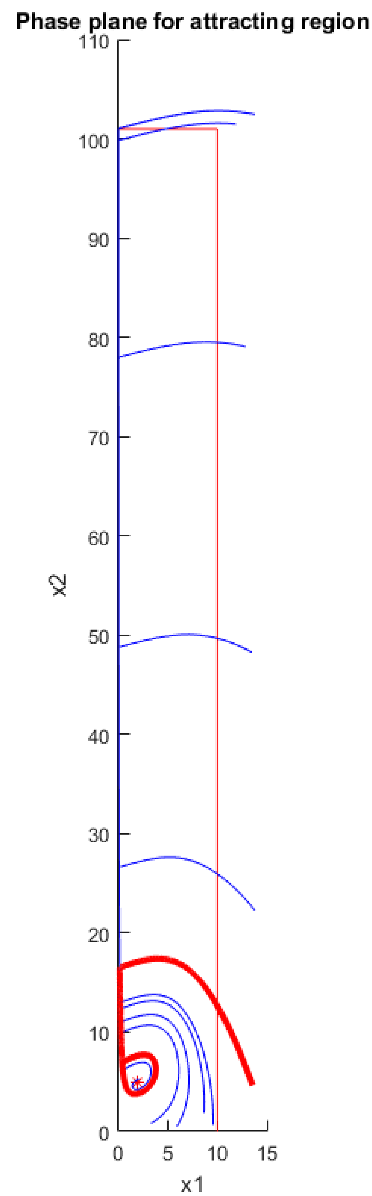
For $x = 0$ (on the y - axis) we have $\dot{x} = 10 > 0$.It implies that trajectories go inside the first quadrant through the y - axis.

For $y = 0$ and $x > 0$ (on the x - axis) we have $y' = x > 0$.

Observe that $\dot{x} < 0$ for $x > 10$, $y > 0$. It implies that the stripe $y \geq 0$, $x \in [0, 10]$ is positively invariant.

Then for $y > 0$ also that $\dot{y} < 0$ for $y > 1 + x^2$ and $x > 0$;.

It implies that the rectangle $[0, 10] \times [0, 101]$ is a positively invariant compact set. Excluding a small open set H_ε containing the equilibrium point $(2, 5)$ and small diameter ε we get a positively invariant compact set $[0, 10] \times [0, 101] \backslash H_\varepsilon$ without equilibrium points that according to the Poincare Bendixson theorem must include at least one periodic orbit because each trajectory starting in this set has a non-empty ω - limit set that is a periodic orbit. So in principle there can be several periodic orbits surrounding this equilibrium point.


Phase plane for attracting region
110
100
90
80
70
60
50
40
30
20
10
0
x2
0
5
10
15
x1

Example. 3.9.1 (from A-P)

Show that the following equation has a limit cycle (a periodic orbit that is an ω - limit set of at least one solution)

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + x_2\left(1 - 3x_1^2 - 2x_2^2\right) \end{aligned}$$

write the system in polar coordinates:

$$\begin{aligned} r &= r\sin^2\theta\left(1 - 3r^2\cos^2\theta - 2r^2\sin^2\theta\right) \\ \theta' &= -1 + \frac{1}{2}\sin(2\theta)\left(1 - 3r^2\cos^2\theta - 2r^2\sin^2\theta\right) \end{aligned}$$

a) Observe that with $r = 1/2$

$$r' = \frac{1}{4}\sin^2\theta\left(1 - \frac{1}{2}\cos^2\theta\right) \geq 0$$

with equality only at $\theta = 0$ and π . Thus $\{x : r > 1/2\}$ is positively invariant (trajectories do not enter the circle $r < 1/2$.

b) The same equation for r' implies that

$$r' \leq r\sin^2\theta\left(1 - 2r^2\right)$$

Thus the annulus $C = \left\{x : 1/2 < r < 1/\sqrt{2}\right\}$ is positively invariant. The only fixed point to the system is outside this annulus. Therefore here is at least one periodic orbit in C that is an ω limit set for all trajectories starting in C (and therefore is a limit cycle).

The solution above used in fact a test function $V(x_1, x_2) = r(x_1, x_2)$.

One could instead of the analytical approach shown above, use a more sofisticated argument, based on considering the better adopted test function $V(x_1, x_2) = 3x_1^2 + 2x_2^2$ *having ellipses as level sets curves: const* $= 3x_1^2 + 2x_2^2$.

Exercise 4.21, p. 158

Consider the system $z' = f(z_1, z_2)$:

$$\begin{aligned} z_1' &= z_2 + z_1 g(z_1, z_2) \\ z_2' &= -z_1 + z_2 g(z_1, z_2) \\ g(z_1, z_2) &= 3 + 2z_1 - z_1^2 - z_2^2 \end{aligned}$$

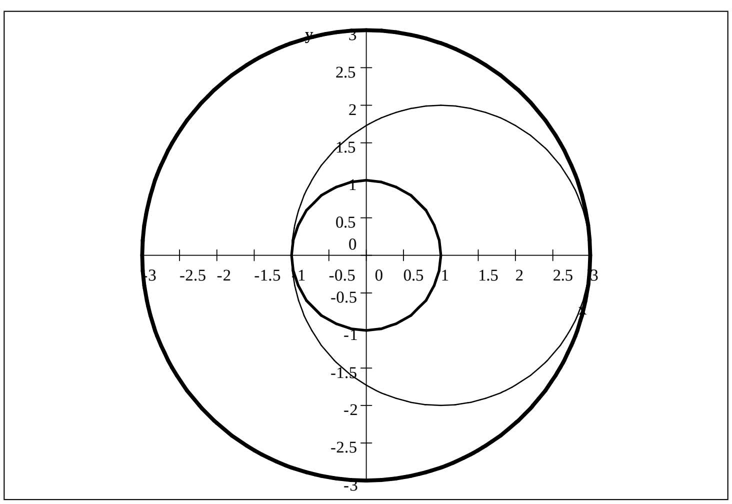
Prove that the system has at least one periodic solution.

Solution.

Consider the test function $V(z_1, z_2) = \left(\frac{z_1^2+z_2^2}{2}\right)$. It's level sets are circles around the origin.

$$\begin{aligned} \frac{dV(z_1(t), z_2(t))}{dt} &= V_f(z_1, z_2) = \nabla V \cdot f(z_1, z_2) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \cdot \begin{bmatrix} z_2 + z_1 g(z_1, z_2) \\ -z_1 + z_2 g(z_1, z_2) \end{bmatrix} \\ &= \left(z_1^2 + z_2^2\right) g(z_1, z_2) = \left(z_1^2 + z_2^2\right)\left(3 + 2z_1 - z_1^2 - z_2^2\right) \\ &= r^2(4 - (1 - z_1)^2 - z_2^2) \end{aligned}$$

The circle $4 = (1 - z_1)^2 + z_2^2$ has center in the point $(1, 0)$ and radius 2



Inside this circle $4-(1 - z_1)^2-z_2^2 > 0$ and therefore $\nabla V \cdot f(z_1, z_2) > 0$. Outside this circle $4 - (1 - z_1)^2 - z_2^2 < 0$ and therefore $\nabla V \cdot f(z_1, z_2) < 0$. Therefore as it is easy to see from the picture, $\nabla V \cdot f(z_1, z_2) \geq 0$ on the circle $z_1^2 + z_2^2 = 1$ with center in the origin beause it is contained inside the circle $4 = (1 - z_1)^2 + z_2^2$, and $\nabla V \cdot f(z_1, z_2) \leq 0$ on the circle $z_1^2 + z_2^2 = 9$ with center in the origin beause it is situated outside the circle $4 = (1 - z_1)^2 + z_2^2$, .

The ring shaped set C: $1 \leq r \leq 3$ is a positively invariant compact set.

The origin is the only equilibrium point for the system, because from the expression $V_f(z_1, z_2) = \nabla V \cdot f(z_1, z_2) = r^2 g(z_1, z_2)$ it follows that other equilibrium points must be situated on the circle $g(z_1, z_2) = 0 = 4 - (1 - z_1)^2 - z_2^2$. Substitution $g(z_1, z_2) = 0$ into the system leads to the conclusion that there are no equilibrium points on this circle.

Therefore the Poincare Bendixson theorem implies that there exists at least one periodic orbit contained in the ring shaped set C.

Exercise. 3.8.2.

Solve a similar problem for the function $g(z_1, z_2) = 3 + z_1 z_2 - z_1^2 - z_2^2$.

Example 3.8.2. Find the limit cycles in the following systems and give their types:

$$\text{(a)}\ \dot{r} = r(r-1)(r-2), \quad \dot{\theta} = 1; \tag{3.67}$$

$$\text{(b)}\ \dot{r} = r(r-1)^2, \quad \dot{\theta} = 1. \tag{3.68}$$

Solution

(a) There are closed trajectories given by

$$r(t) \equiv 1, \quad \theta = t \quad \text{and} \quad r(t) \equiv 2, \quad \theta = t. \tag{3.69}$$

Furthermore

$$\dot{r} \begin{cases} > 0, & 0 < r < 1 \\ < 0, & 1 < r < 2 \\ > 0, & r > 2 \end{cases}. \tag{3.70}$$

The system therefore has two circular limit cycles: one stable ($r = 1$) and one unstable ($r = 2$).

(b) System (3.68) has a single circular limit cycle of radius one. However, $\dot{r}$ is positive for $0 < r < 1$ *and* $r > 1$, so the limit cycle is semistable. □

Example. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = x - 2y - x\left(2x^2 + y^2\right) \\ y' = 4x + y - y\left(2x^2 + y^2\right) \end{cases} \qquad \textbf{(4p)}$$

Solution. Consider the following test function: $V(x, y) = 2x^2 + y^2$. Denoting the right hand side in the equation by vectorfunction $F(x, y)$ we conclude that

$V_f = \nabla V \cdot f = 4x^2 - 8xy - 4x^2\left(2x^2 + y^2\right) + 8xy + 2y^2 - 2y^2(2x^2 + y) = 2\left(1 - \left(2x^2 + y^2\right)\right)\left(2x^2 + y^2\right)$.

It implies that the elliptic shaped ring: $R = \{(x, y) : 0.5 \leq (2x^2 + y) \leq 2\}$ is a positive invariant compact set for the ODE, because velocity vectors are directed inside of this ring both on it's outer and inner boundaries ($\nabla V \cdot F < 0$ for $(2x^2 + y) = 2$ and $\nabla V \cdot F > 0$ for $(2x^2 + y) = 0.5$.

The origin is the only equilibrium point of the system**. It is not so easy to see from the system of equations itself.** But one can see it easier by cheching first zeroes of $V_f(x, y)$ that is a scalar function and evidently must be zero in all equilibrium points..

We observe that $V(x, y) = 2x^2 + y^2$ is positive definite and $\nabla V \cdot f(x, y) = 0$ only if $(x, y) = (0, 0)$ or if $(2x^2 + y^2) = 1$.But it is easy to see from the expression for the right hand side for the ODE that in the last case (x, y) cannot be equilibrium point because the right hand side becomes linear with nondegenerate matrix and is zero only in the origin $(x, y) = (0, 0)$. The equation for equilibrium points on the level set $(2x^2 + y^2) = 1$ is the following:

$$\begin{cases} 0 = x - 2y - x = -2y \\ 0 = 4x + y - y = 4x \end{cases}$$

By the Poincare-Bendixson theorem the positively invariant set R not including any equilibrium point must include at least one orbit of a periodic solution.■

Generalized Poincare-Bendixson's theorem.

The following theorem gives a more complete description of the types of ω - limit sets in the plane $\mathbb{R}^2$.

Theorem (generalized Poincare-Bendixson)

Let M be an open subset of $\mathbb{R}^2$ and $f : M \to \mathbb{R}^2$ and $f \in C^1$. Fix $\xi \in M$ and suppose that $\Omega(\xi) \neq \emptyset$, compact, connected and contains only finitely many equilibrium points.

Then one of the following cases holds:

(i) $\Omega(\xi)$ is an equilibrium point

(ii) $\Omega(\xi)$ is a periodic orbit

(iii) $\Omega(\xi)$ consists of finitely many fixed points $\{x_j\}$and non-closed orbits γ such that ω and α - limit points of γ belong to $\{x_j\}$.

Example. While we have already seen examples for case (i) and (ii) in the Poincaré–Bendixson theorem we have not seen an example for case (iii). Hence we consider the vector field

$$f(x,y) = \begin{pmatrix} y + x^2 - \alpha x(y - 1 + 2x^2) \\ -2(1+y)x \end{pmatrix}.$$

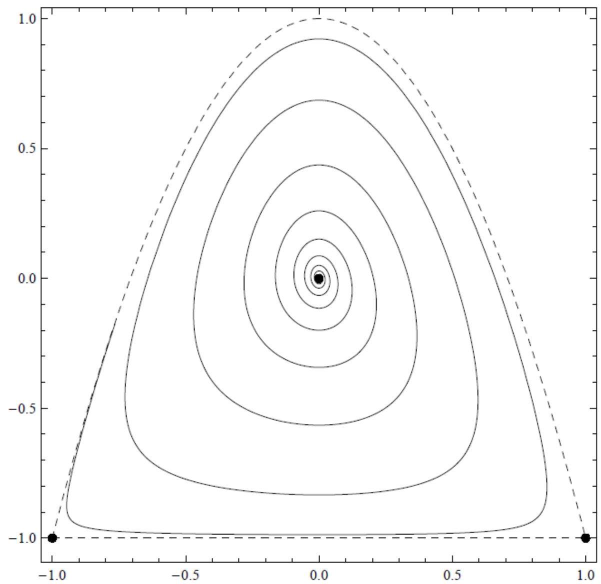
First of all it is easy to check that the curves $y = 1 - 2x^2$ and $y = -1$ are invariant. Moreover, there are four fixed points $(0,0)$, $(-1,-1)$, $(1,-1)$, and $(\frac{1}{2\alpha}, -1)$. We will chose $\alpha = \frac{1}{4}$ such that the last one is outside the region bounded by the two invariant curves. Then a typical orbit starting inside this region is depicted in Figure 7.9: It converges to the unstable fixed point $(0,0)$ as $t \to -\infty$ and spirals towards the boundary as $t \to +\infty$. In

$V(x,y) = y + 2x^2 - 1$. $V_f(x,y) = 4x\,(y + x^2 - \alpha x(V(x,y))) - 2(1+y)x|_{y=1-2x^2} =$

$= 4x(1-2x^2) + 4x^3 - 2x(2-2x^2) = 4x - 4x + 4x^3 - 4x^3 = 0$

$f_y(x,y)|_{y=-1} = 0$

$A(0,0) = \begin{bmatrix} \alpha & 1 \\ -2 & 0 \end{bmatrix}$; $\det A = 2$; $\mathrm{tr}A = \alpha > 0$. Therefore the equilibrium point $(0,0)$ is a repeller, unstable spiral.



particular, its $\omega_+((x_0, y_0))$ limit set consists of three fixed points plus the orbits joining them.

To prove this consider $H(x,y) = x^2(1+y) + \frac{y^2}{2}$ and observe that its change along trajectories

$$\dot{H} = 2\alpha(1 - y - 2x^2)x^2(1+y)$$

is nonnegative inside our region (its boundary is given by $H(x,y) = \frac{1}{2}$). Hence it is straightforward to show that every orbit other than the fixed point $(0,0)$ converges to the boundary. $\diamond$