May 19, 2022

Lecture notes on general and periodic linear ODEs

Plan for the chapter

1. Transition matrix function, existence and equations. Lemma 2.1, p.24, Cor. 2.3, p.26.

2. Uniqueness of solutions and dimenstion of solution space. Th. 2.5, p. 28, Prop. 2.7(1), p.30

3. Group properties of transition matrix function and transition mapping.

(Chapman - Kolmogorov relations) Cor. 2.6, p. 29

4. Fundamental matrix solution and its connection with transition matrix function. Prop. 2.8, p.33

5. Transition matrix function for periodic linear systems. Formula. 2.31, p. 45.

7. Monodromy matrix and properties of transition matrix function for periodic systems. Th. 2.30, p. 53.

8. Logarithm of a matrix. Prop. 2.29, p. 53.

9. Floquet multipliers and exponents.

10. Boundedness and zero limits for solutons to periodic linear systems. Th. 2.31, p.54. Cor. 2.33, p-59

11. Existence of periodic solutions to periodic linear systems. Prop. 2.20, p. 45

12. Hill equation and Kapitza pendulum. pp. 55-57.

13. Abel-Liouville's formula. Prop. 2.7(2)., p.30.

14. Conditions for existence of unbounded solutions based on the Abel-Liouvilles formula

Lecture 19

0.1 Transition matrix function for non-autonomous linear systems.

The subject of this chapter of lecture notes is general non - autonomous linear systems of ODEs and in particular systems with periodic coefficients and Floquet theory for them.

The general theory for non - autonomous linear systems (linear systems with variable coefficients) is very similar to one for systems with constant coefficients. The existence follows from Picard - Lindelöf theorem or can be established through the solution of the integral form of equations by iterations. Uniqueness follows general results for non-linear systems with right hand side $f(t,x)$ locally Lipschits with respect to x and is based on the Grönwall inequality. These results lead to the fundamental result on the dimension of the space of solutions that is based on the uniqueness result similarly to the proof for systems with constant coefficients.

The essential difference from the case with constant coefficients is that in the case with variable coefficients one cannot find analytical solutions except some particular cases as systems with triangular matrices.

We consider the I.V.P. in the differential

$$x' = A(t)x(t), \quad x(\tau) = \xi \tag{1}$$

or in the integral form

$$x(t) = \xi + \int_{\tau}^{t} A(s)x(s)ds \tag{2}$$

with matrix valued function $A : J \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$) that is continuous or piecewise continuous on the interval J.

Here it is important that the initial time τ is an arbitrary real number from J, not just zero.

The solution is defined as a continuous function $x(t)$ on an interval I that includes point $\tau \in I$, acting into $\mathbb{R}^N$ or $\mathbb{C}^N$, and satisfying the integral equation (2). By a version of Calculus main theorem (Newton-Leibnitz theorem) the solution defined in such a way will satisfy the differential equation (1) in points t where $A(t)$ is continuous.

We remind the following lemma considered in the beginning of the course.

Lemma. The set of solutions $\mathcal{S}_{\text{hom}}$ to (2) is a linear vector space.

□

It motivates us to search solution in the form

$$x(t) = \Phi(t,s)\xi$$

where $\Phi(t,s)$ is a continuous matrix valued function on $J \times J$ and ξ is an arbitrary initial data at $t = s$: $x(s) = \xi$. It implies also that $\Phi(s,s) = I$.

Definition. The matrix $\Phi(t,\tau)$ is called **transition matrix function.**

Existence and uniqueness of solutions to I.V.P.

Theorem 2.5, p. 28 L&R

Let $(\tau,\xi) \in J \times \mathbb{R}^N (J \times \mathbb{C}^N)$. The function $x(t) = \Phi(t,\tau)\xi$ is a unique solution to the I.V.P. (1).If $y : J_y \to \mathbb{R}^N$ or $(\mathbb{C}^N)$ is a another solution to (1) $x' = A(t)x(t)$, then $y(t) = x(t)$ for all $t \in J_y$.

Proof.

The existence of solutions to (1) follows from the general theorem about uniqueness of solutions to ODEs with the right hand side $f(t,x)$ that is continuous and locally Lipschitz continuous with respect to the space variable x. Lipschitz constant can be chosen in this case as

$$\begin{aligned} L &= \sup_{t\in[a,b]} \|A(t)\| \\ \sup_{t\in[a,b]} \|A(t)x - A(t)y\| &= \sup_{t\in[a,b]} \|A(t)(x-y)\| \le \sup_{t\in[a,b]} \|A(t)\| \, \|x-y\| \end{aligned}$$

The maximal interval $I_\xi = J$, by Proposition 4.12, p. 114, because the right hand side $A(t)x$ in the equation rises linearly on any compact interval $[a,b] \subset J$:

$$\sup_{t\in[a,b]} \|A(t)x\| \leq \sup_{t\in[a,b]} \|A(t)\| \, \|x\|$$

The existence of the **transition matrix** $\Phi(t,\tau)$ is shown in the following way.

Substituting the expression $x(t) = \Phi(t,s)\xi$ into the integral form of the i.V.P., we arrive to the vector equation

$$\begin{aligned}
\Phi(t,s)\xi &= \xi + \int_s^t A(\sigma)\Phi(\sigma,s)\xi d\sigma \Longrightarrow \\
\Phi(t,s)\xi &= \left(I + \int_s^t A(\sigma)\Phi(\sigma,s)d\sigma\right)\xi
\end{aligned}$$

with arbitrary $\xi \in \mathbb{R}^N$ that implies the matrix equation for $\Phi(t,s)$:

$$\Phi(t,s) = I + \int_s^t A(\sigma)\Phi(\sigma,s)d\sigma \tag{3}$$

or the same equation in differential form valid outside points of disconituity of $A(t)$:

$$\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s); \quad \Phi(s,s) = I.$$

This matrix equation is equivalent to n similar I.V.P. for ODEs

$$\varphi_k'(t,s) = A(t)\varphi_k(t,s); \quad \varphi_k(s,s) =$$

for columns $\varphi_k(t,s)$, $k = 1,...n$ of the matrix $\Phi(t,s)$ with initial conditions that are corresponding columns from the unit matrix I. These equations have unuque solutions with maximal interval $I_{\max} = J$, by previous standard arguments for this equation: Picard-Lindelöf theorem and a proposition about the extension of solutions for the right hand side $f(t,x)$ in the ODE $x' = f(t,x)$, rising not faster then linearly in x variable on any compact time interval.

We can also show the existence of solution to this integral equation (3) for $\Phi(t,s)$ directly by means of iterational approximations $M_n(t,s)$ to $\Phi(t,s)$ introduced in the following way:

$$M_1(t,s) = I; \quad M_{n+1}(t,s) = I + \int_s^t A(\sigma)M_n(\sigma,s)d\sigma, \quad \forall n \in \mathbb{N} \tag{4}$$

Lemma 2.1 , p. 24 and **Corollary 2.3,** p. 26 in L&R

For any closed and bounded interval $[a,b] \subset J$ the sequence $\{M_n(t,s)\}$ converges uniformly on $[a,b] \times [a,b]$ to a continuous on $[a,b] \times [a,b]$ matrix valued function $\Phi(t,s)$ that satisfies the integral equation (3).

Point out that outside of points of discontinuity of the matrix $A(t)$ the function $\Phi(t,s)$ satisfies the differential equation $\frac{d}{dt}\Phi(t,s) = A(t)\Phi(t,s)$.

The proof of the Lemma can be found in lecture note from 2021 available in the Modulus Lecture notes in Canvas and in the course book.

The product $x(t) = \Phi(t,\tau)\xi$ gives by construction the solution to I.V.P. to the equation $x'(t) = A(t)x(t)$ with initial data $x(\tau) = \xi$. In the case when $A(t)$ is only piecewise continuous, $x(t)$ will be continuous and satify the corresponding integral equation. It satifies the differential equation outside discontinuities of $A(t)$.

Example. For an autonomous linear system with constant matrix A the transition matrix function is $\Phi(t,\tau) = \exp(A(t-\tau))$.

0.2 Solution space.

We have considered a particular variant of the following theorem in the case of linear systems of ODEs with constant coefficients. The formulation and the proof we suggested are based only on the fact that the set of solutions $\mathbb{S}_h$ is a linear vector space and on the property of the uniquness of solutions. We repeat this argument here again for convenience with some corollaries about the structure of the transition matrix $\Phi(t, \tau)$.

Proposition 2.7 (1), p.30, L&R.

Let $b_1, ..., b_N$ be a basis in $\mathbb{R}^N$ (or $\mathbb{C}^N$) and let $\tau \in J$.

Let $\Phi(t, \tau)$ be a transition matrix to the equation

$$x' = A(t)x$$

with $A(t)$ being a matrix valued function $A : J \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$), piecewise continuous on the interval J.

Then functions $y_j : J \to \mathbb{R}^N$ (or $\mathbb{C}^N$) defined as solutions

$$y_j(t) = \Phi(t, \tau)b_j$$

with $j = 1, ..., N$ to , the equation above form a basis of the solution space $\mathbb{S}_h$ of the equation.

In particular $\mathbb{S}_h$ is N -dimensional and for every solution $x(t) : J \to \mathbb{R}^N$ (or $\mathbb{C}^N$) there exist scalars $\gamma_1, ...\gamma_N$ such that

$$x(t) = \sum_{j=1}^{N} \gamma_j y_j(t)$$

for all $t \in J$.

Proof

We can just repeat here the proof that we gave earlier. Point out that it is more general than one given in the course book.

Suppose that at some time t solutions $y_j(t)$ are linearly dependent. It means that there are constants $\{a_j\}_{j=1}^N$ not all zero such that

$$\sum_{j=1}^{N} a_j y_j(t) = 0$$

at this time. On the other hand there is a solution that satisfies this condition. It is zero solution $x_*(t) = 0$ for all t.

But then these two solutions must coinside because solutions are unique!!! Namely $\sum_{j=1}^N a_j y_j(t) = 0$ for all times including $t = \tau$.Therefore $\sum_{j=1}^N a_j y_j(\tau) = \sum_{j=1}^N a_j b_j = 0$ because b_j are initial conditions at $t = \tau$ for y_j. It is a contradiction because vectors b_j , $j = 1, ..., N$ are linearly independent. Therefore $y_j(t)$ with $j = 1, ..., N$ are linearly independent for all t in J. ■

Example.

Calculate the transition matrix function $\Phi(t, s)$ for the system of equations

$$\begin{cases} x_1' = t\,x_1 \\ x_2' = x_1 + t\,x_2 \end{cases}$$

$$\begin{aligned} x' &= A(t)x; \quad A(t) = \begin{bmatrix} t & 0 \\ 1 & t \end{bmatrix} \\ x(\tau) &= \xi \end{aligned}$$

$$x(t) = \Phi(t, \tau)\xi$$

Here the matrix $A(t)$ is triangular.

The system of ODE above has triangular matrix and can be solved recursively starting from the first equation.

The fundamental matrix $\Phi(t, \tau)$ satifies the same equation, namely

$$\begin{aligned} \frac{d}{dt}\Phi(t, \tau) &= A(t)\Phi(t, \tau) \\ \Phi(\tau, \tau) &= I \end{aligned}$$

$\Phi(t, \tau)$ has columns $\pi_1(t, \tau)$ and $\pi_2(t, \tau)$ that at the time $t = \tau$ have initial values $[1, 0]^T$ and $[0, 1]^T$, because $\Phi(\tau, \tau) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

We need to find two solutions $\pi_1(t, \tau)$ and $\pi_2(t, \tau)$ that at the time $t = \tau$ have initial values $[1, 0]^T$ and $[0, 1]^T$ to the equation

$$x' = A(t)x;$$

We will use a general solution to the scalar linear equation $x' = p(t)x + g(t)$ with initial data $x(\tau) = x_0$ calculated using the primitive function $\mathbb{P}(t, \tau) = \int_\tau^t p(s)ds$ of $p(t)$:

$$x(t) = \exp\{\mathbb{P}(t, \tau)\}\, x_0 + \int_\tau^t \exp\{\mathbb{P}(t, s)\}\, g(s)ds$$

A derivation of this formula using the integrating factor idea follows.

$$\begin{aligned}
x' &= p(t)x + g(t), \quad x_0 = x(\tau) \\
\mathbb{P}(t,\tau) &= \int_\tau^t p(s)ds \\
\exp\{-\mathbb{P}(t,\tau)\}\, x' &= \exp\{-\mathbb{P}(t,\tau)\}\, p(t)x + \exp\{-\mathbb{P}(t,\tau)\}\, g(t) \\
\exp\{-\mathbb{P}(t,\tau)\}\, x' - p(t)\exp\{-\mathbb{P}(t,\tau)\}\, x &= \exp\{-\mathbb{P}(t,\tau)\}\, g(t) \\
\exp\{-\mathbb{P}(t,\tau)\}\, x' + (\exp\{-\mathbb{P}(t,\tau)\})'\, x &= \exp\{-\mathbb{P}(t,\tau)\}\, g(t) \\
[\exp\{-\mathbb{P}(t,\tau)\}\, x]' &= \exp\{-\mathbb{P}(t,\tau)\}\, g(t) \\
\int_\tau^t [\exp\{-\mathbb{P}(s,\tau)\}\, x(s)]'\, ds &= \int_\tau^t \exp\{-\mathbb{P}(s,\tau)\}\, g(s)ds \\
\exp\{-\mathbb{P}(t,\tau)\}\, x(t) - \exp\{-\mathbb{P}(\tau,\tau)\}\, x_0 &= \int_\tau^t \exp\{-\mathbb{P}(s,\tau)\}\, g(s)ds \\
\exp\{-\mathbb{P}(t,\tau)\}\, x(t) - \exp\{0\}\, x_0 &= \int_\tau^t \exp\{-\mathbb{P}(s,\tau)\}\, g(s)ds
\end{aligned}$$

$$\begin{aligned}
x(t) &= \exp\{\mathbb{P}(t,\tau)\}\, x_0 + \int_\tau^t \exp\{\mathbb{P}(t,\tau)\}\exp\{-\mathbb{P}(s,\tau)\}\, g(s)ds \\
x(t) &= \exp\{\mathbb{P}(t,\tau)\}\, x_0 + \int_\tau^t \exp\{\mathbb{P}(t,\tau) - \mathbb{P}(s,\tau)\}\, g(s)ds \\
\mathbb{P}(t,\tau) - \mathbb{P}(s,\tau) &= \int_\tau^t p(z)dz - \int_\tau^s p(z)dz = \int_\tau^t p(z)dz + \int_s^\tau p(z)dz = \\
\int_s^t p(z)dz &= \mathbb{P}(t,s) \\
x(t) &= \exp\{\mathbb{P}(t,\tau)\}\, x_0 + \int_\tau^t \exp\{\mathbb{P}(t,s)\}\, g(s)ds; \\
x(\tau) &= x_0
\end{aligned}$$

In the equation

$$x_1' = t\, x_1$$

the coefficient $p(t) = t$, therefore $\mathbb{P}(t,\tau) = \int_\tau^t s\, ds = \left(\frac{1}{2}s^2\right)\Big|_\tau^t = \frac{1}{2}\left(t^2 - \tau^2\right)$ and the solution

$$x_1(t) = \exp(\frac{1}{2}\left(t^2 - \tau^2\right))x_1(\tau).$$

The second equation

$$x_2' = t\, x_2 + x_1$$

is similar but inhomogeneous:

$$x_2(t) = \exp(\mathbb{P}(t,\tau))x_2(\tau) + \int_\tau^t \exp(\mathbb{P}(t,s))x_1(s)ds.$$

Substituting $\mathbb{P}(t,\tau) = \frac{1}{2}\left(t^2 - \tau^2\right)$ we conclude that $\exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau) + \int_\tau^t \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_1(\tau)ds$

$$\begin{aligned} x_2(t) &= \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau) + \int_\tau^t \exp(\frac{1}{2}\left(t^2-s^2\right))\exp(\frac{1}{2}\left(s^2-\tau^2\right))x_1(\tau)ds \\ &= \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau) + \int_\tau^t \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_1(\tau)ds \end{aligned}$$

And

$$x_2(t) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))x_2(\tau) + \exp(\frac{1}{2}\left(t^2-\tau^2\right))(t-\tau)x_1(\tau).$$

The fundamental matrix solution $\Phi(t,\tau)$ has columns that are solutions to $x' = A(t)x$ with initial data - that are columns in the unit matrix: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

Taking $x_1(\tau) = 1$ and $x_2(\tau) = 0$ we get $x_1(t) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))$ with $x_2(t) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))(t-\tau)$

Taking $x_1(\tau) = 0$ and $x_2(\tau) = 1$ we get $x_1(t) = 0$ with $x_2(t) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))$ and the fundamental matrix solution in the form

$$\Phi(t,\tau) = \exp(\frac{1}{2}\left(t^2-\tau^2\right))\begin{bmatrix} 1 & 0 \\ t-\tau & 1 \end{bmatrix}$$

0.3 Group properties of transition matrix. Chapman - Kolmogorov relations.

The transition matrix $\Phi(t,\tau)$ defines a simple expression for the **transition mapping** $\varphi(t,\tau,\xi)$, that maps initial data ξ at time τ into the value of the solution of I.V.P. $\varphi(t,\tau,\xi) = x(t) = \Phi(t,\tau)\xi$ of the system at time t.

In the case of linear systems a simpler expression for translation properties of the transition mapping can be formulated.

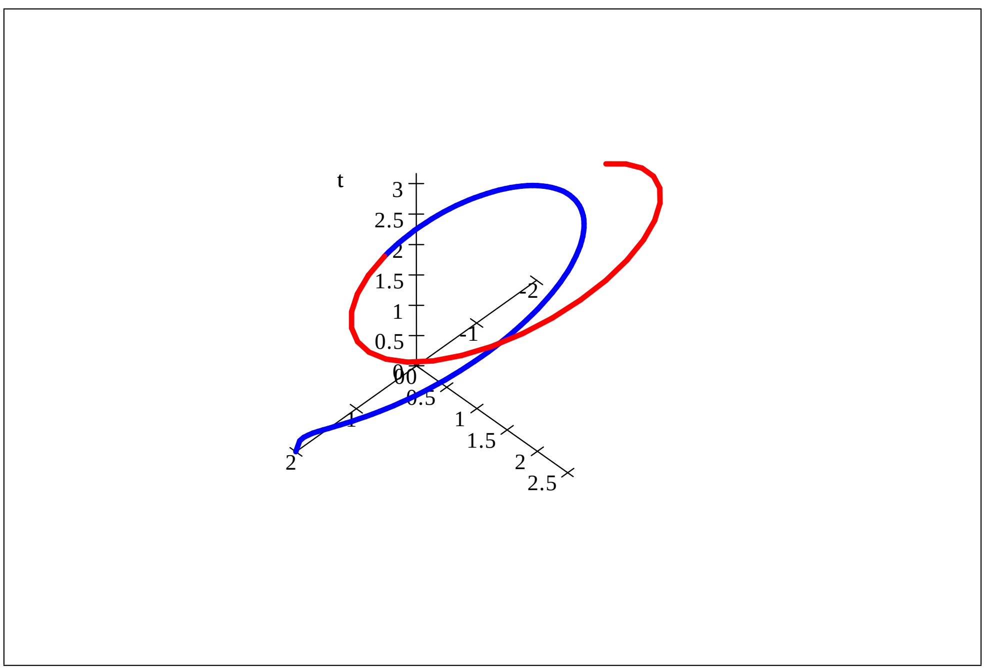
$$\Phi(t,\tau)\xi = \Phi(t,\sigma)\left[\Phi(\sigma,\tau)\xi\right]$$

The proof is repeated here for convenience.

Let us consider two consequtive solutions $x(t) = \Phi(t,\tau)\xi$ and $y(t) = \Phi(t,\sigma)\left(\Phi(\sigma,\tau)\xi\right)$ of the equation $x' = A(t)x(t)$, that continue each other in the time point $t = \sigma$ where the second solution $y(t)$ attains the initial state taken in the point where the first solution $x(t)$ arrives at time $t = \sigma$.

Together with the uniquness of solutions, this consideration leads to the *group property* of the transition mapping and the transition matrix. The group property means that moving the system governed by the equation $x'(t) = A(t)x(t)$ from time τ to time t is the same as to move it first from time τ to time σ (blue curve) and then to move it without break from time σ to time t (red curve)

$$\Phi(t,\tau)\xi = \Phi(t,\sigma)\left[\Phi(\sigma,\tau)\xi\right]$$



Point out that these two "movements" do not need to go both in the positive direction in time as it is in the picture. One of these movements (or both) can go backward in time. Another observation is that the linearity of the system was not essential for this reasoning, only the uniqueness of solutions.

We have proven (almost) the following theorem.

Corollary 2.6, p.29, L&R (Chapman - Kolmogorov relations)

For all $t,\, \sigma,\, \tau \in J$

$$\begin{aligned} \Phi(t,\tau) &= \Phi(t,\sigma)\Phi(\sigma,\tau), && (5)\\ \Phi(t,t) &= I, \\ \Phi(\tau,t)\Phi(t,\tau) &= \Phi(\tau,\tau) = I \\ \Phi(\tau,t) &= (\Phi(t,\tau))^{-1} && (6) \end{aligned}$$

Proof.

The first statement has been proven already. The second follows from the integral equation for the transition matrix. The third one follows from the first two. We apply the first statement $\Phi(t,\tau)\ \Phi(\tau,t) = \Phi(t,t) = I$ therefore $\Phi(\tau,t)$ is the right inverse of $\Phi(t,\tau)$. The same argument for this expression with t and τ changed their roles leads to that $\Phi(\tau,t)$ is the left inverse of $\Phi(t,\tau)$.■

Example.

Remeber that in the case with autonomous systems $x' = Ax$, the transition matrix is $\Phi(t,\tau) = \exp\left((t-\tau)A\right).$

Therefore in this case the Chapman - Kolmogorov relations follow from properties of matrix exponent: $\exp\left\{(t-\tau)A\right\} = \exp\left\{(t-\sigma)A\right\}\exp\left\{(\sigma-\tau)A\right\}$

0.4 Non-homogeneous linear systems.

We consider the I.V.P. for non-homogeneous linear system

$$x'(t) = A(t)x(t) + b(t), \quad x(\tau) = \xi, \quad (\tau,\xi) \in J \times \mathbb{R}^N (J \times \mathbb{C}^N)$$

We suppose here that $A : J \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$) is continuous or piecewise continuous and denote by $\Phi(t,\tau)$ the transition matrix function generated by $A(t)$. We rewrite the I.V.P. for the system also in integral form

$$x(t) = \xi + \int_\tau^t \left(A(\sigma)x(\sigma) + b(\sigma)\right) d\sigma,$$

that allows to consider continuous solutions in the case when A is only peacewise continuous. In this case solutions satisfy the differential form of the problem in between of discontinuties of A.

Theorem 2.15, p. 41 L&R

Let $(\tau,\xi) \in J \times \mathbb{R}^N$.The function

$$x(t) = \Phi(t,\tau)\xi + \int_\tau^t \Phi(t,\sigma)b(\sigma)d\sigma,$$

is a unique solution to the I.V.P. above.

Proof can be found in lecture note from 2021 or in the course book.

1 Systems with periodic coefficients: Floquet theory

We consider here linear homogeneous systems of ODE's with $J = R$ and a continuous or piecewise continuous matrix $A : \mathbb{R} \to \mathbb{R}^{N\times N}$ (or $\mathbb{C}^{N\times N}$), with period $p > 0$:

$$x'(t) = A(t)x(t), \quad A(t+p) = A(t), \quad \forall t \in \mathbb{R}$$

Let Φ be a transition matrix generated by a periodic $A(t)$.

Shifting invariance property.(formula 2.31, p. 45 in L.R.)

We are going to prove an important *shifting invariance property* of this transition matrix function, namely that

$$\Phi(t+p, \tau+p) = \Phi(t, \tau) \tag{7}$$

Structure of the transition matrix for a time interval including a finite number of periods. (formula 2.32, p. 45 in L.R.)

(Motivation to introducing the monodromy matrix)

Another property specifying further how the periodicity of the system influences the structure of the transition matrix $\Phi(t, \tau)$ is expressed by the following relations:

$$\Phi(t+p, \tau) = \Phi(t, 0)\, [\Phi(p, 0)]\, \Phi(0, \tau) \tag{8}$$

$$\Phi(t+n\,p, \tau) = \Phi(t, 0)\, [\Phi(p, 0)]^n\, \Phi(0, \tau) \tag{9}$$

for any $(t, \tau) \in \mathbb{R} \times \mathbb{R}$.

Definition of the Monodromy matrix

The transition matrix $\Phi(p, 0)$ for a periodic linear system with period p is called the **monodromy matrix** (this standard notion is not used in the course book)

Proof of the shifting invariance property.

This first property becomes untuitively clear after followig arguments:

The matrix $\Phi(t, \tau)$ satisfies the equation

$$\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau)$$

with initial condition , $\Phi(t, \tau)|_{t=\tau} = I$.

The matrix $\Phi(t+p, \tau+p)$ satisfies the equation

$$\frac{\partial}{\partial t}\Phi(t+p, \tau+p) = A(t+p)\Phi(t+p, \tau+p)$$

with initial condition , $\Phi(t+p, \tau+p)|_{t=\tau} = I$.

Now we observe (!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!)

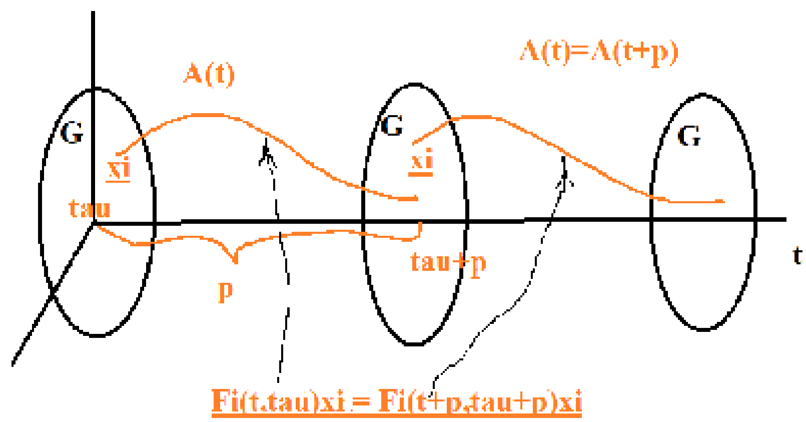
that $A(t) = A(t+p)$. Substituting it in the second equation we get the equation

$$\frac{\partial}{\partial t}\Phi(t+p,\tau+p) = A(t)\Phi(t+p,\tau+p)$$

with the same initial condition, $\Phi(\tau+p,\tau+p) = I$ on the interval $t \in [\tau, t)$.

It implies that $\Phi(t,\tau)$ and $\Phi(t+p,\tau+p)$ satisfy in fact the same equation with the same initial conditions $\Phi(t+p,\tau+p)|_{t=\tau} = I$. The uniqueness of solutions implies that they must be equal: $\Phi(t+p,\tau+p) = \Phi(t,\tau)$.

A prove using the integral form of the equation is presented in the course book.■



Proof of the structure formula for the transition matrix for a periodic system

The proof is based on a combination of the shifting property with the Chapman-Kolmogorov relations.

$$\begin{aligned}&\Phi(t+p,\tau) \overset{Ch.-Kol.}{=} \Phi(t+p,\tau+p)\Phi(\tau+p,\tau) \overset{Shift}{=} \Phi(t,\tau)\Phi(\tau,\tau-p)\\ &\overset{Ch.-Kol.}{=} \Phi(t,\tau)\Phi(\tau,0)\Phi(0,\tau-p) \overset{Ch.-Kol.\_and\_Shift}{=} \Phi(t,0)\Phi(p,\tau)\\ &\overset{Ch.-Kol.}{=} \Phi(t,0)\Phi(p,0)\Phi(0,\tau)\end{aligned}$$

The second equality for the shift $n\,p$ in n periods p in time is derived by the repetition of the last argument and induction

$$\begin{aligned}&\Phi(t+np,\tau) \overset{Ch.-Kol.}{=} \Phi(t+np,\tau+np)\Phi(\tau+np,\tau) \overset{Shift}{=} \Phi(t,\tau)\Phi(\tau,\tau-np)\\ &\overset{Ch.-Kol.}{=} \Phi(t,\tau)\Phi(\tau,0)\Phi(0,\tau-np) \overset{Ch.-Kol.}{=} \Phi(t,0)\Phi(np,\tau)\\ &\overset{Ch.-Kol.}{=} \Phi(t,0)\Phi(np,0)\Phi(0,\tau)\end{aligned}$$

and from the observation that

$$\Phi(np,0) \overset{Ch.-Kol\_n\_times}{=} \Phi(np,np-p)...\Phi(kp,kp-p)...\Phi(2p,p)\Phi(p,0) \overset{shifting\_for\_each\_term}{=} [\Phi(p,0)]^n$$

that follows from the Chapman-Kolmogorov relation and from the fact that $\Phi(t,0)$ satisfies the same equation on each interval $[kp,(k+1)\,p]$, (shift invariance property) because $A(t) = A(t+p)$ is a periodic matrix with period p. It implies the desired formula:

$$\Phi(t + n\,p, \tau) = \Phi(t, 0)\,[\Phi(p, 0)]^n\,\Phi(0, \tau)$$

Lecture 20
Summary
on transition matrix for periodic systems

1. Shifting property of transition matrix for periodic systems:

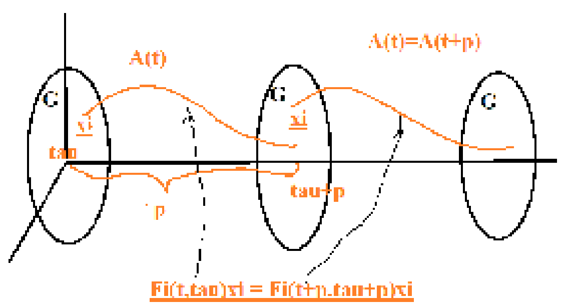
$$\Phi(t + p, \tau + p) = \Phi(t, \tau)$$

2. Structure formula for transition matrix of periodic systems

$$\Phi(t + p, \tau) = \Phi(t, 0)\,[\Phi(p, 0)]\,\Phi(0, \tau) \tag{10}$$

$$\Phi(t + n\,p, \tau) = \Phi(t, 0)\,[\Phi(p, 0)]^n\,\Phi(0, \tau) \tag{11}$$

3. Picture of the transition mapping $\varphi(t, \tau, \xi) = \Phi(t, \tau)\xi$ generated by the transition matrix $\Phi(t, \tau)$ to a differential equation $x'(t) = A(t)x(t)$, with periodic coefficients $A(t + p) = A(t)$



Example illustrating ideas of Floquet theory on a scalar linear equation.

Consider the following scalar linear equation with periodic coefficient $A(t) = (\sin(4t) - 0.1)$ with period $p = 0.5\pi$:

$$\frac{dx}{dt} = (\sin(4t) - 0.1)\,x,$$

We will find the monodromy matrix for this simple equation and demonstrate all objects related to the Floquet theorem that follows.

The exact general solution is:

$$x(t) = C\exp\left(-0.25\cos\left(4t\right) - 0.1t\right)$$

with arbitrary constant C, can be found by the method with integrating factor.

$-0.25\cos(4t) - 0.1t$ is the primitive function of the coefficient $(\sin(4t) - 0.1)$ in front of x in the equation.

To find the solution equal to 1 at $t = 0$ that is the transition "matrix" in the scalar case, we calculate the expression $\exp(-0.25\cos(4.0t))\, e^{-0.1t}\big|_{t=0} = 0.778\,8$ and choose $C = \frac{1}{0.778\,8}$ in the expression for the general solution $x(t)$.

The transition "matrix" is:

$$\Phi(t, 0) = \frac{1}{0.778\,8} \exp(-0.25\cos(4.0t))\, e^{-0.1t}$$

The period of the coefficient in the system is $p = 0.5\pi$ and the **monodromy matrix** is $\Phi(p, 0) = \Phi(0.5\pi, 0)$:

$\Phi(p, 0) = \frac{1}{0.778\,8} \exp(-0.25\cos(4.0t))\, e^{-0.1t}\Big|_{t=0.5\pi} = 0.854\,64$

The eigenvalue μ of the (1x1) "monodromy matrix" $\Phi(p, 0)$ coinsides with it's value: $\mu = 0.854\,64 < 1$ and is strictly less than 1.

Consider the logarithm $G = \ln(\Phi(p, 0))$ of the **monodromy matrix** $\Phi(p, 0)$:

$$G = \ln(\Phi(p, 0)) = \ln\left(\frac{1}{0.778\,8} \exp(-0.25\cos(4.0t))\, e^{-0.1t}\right)\Bigg|_{t=0.5\pi} = -0.157\,08$$

$F = \frac{G}{p} = \frac{-0.157\,08}{0.5\pi} = -0.1 < 0$

Therefore the eigenvalue $\lambda = -0.1$ of the "matrix"

$$F = \frac{1}{p} G = \frac{1}{p} \ln(\Phi(p, 0))$$

is negative.

The transition matrix to the linear homogeneous "system"

$$y'(1) = F y(t)$$

is

$$\exp(Ft) = \exp(t\frac{G}{p}) = \exp(-0.1t).$$

Compare black and orange graphs for $\exp(t\frac{G}{p})$ and for $\Phi(t, 0) = \frac{1}{0.778\,8} \exp(-0.25\cos(4.0t))\, e^{-0.1t}$. Observe that $\exp(t\frac{G}{p})$ and $\Phi(t, 0)$ coinside in points $t = pn = (0.5\pi)\, n$, $n = 1, 2, 3...$

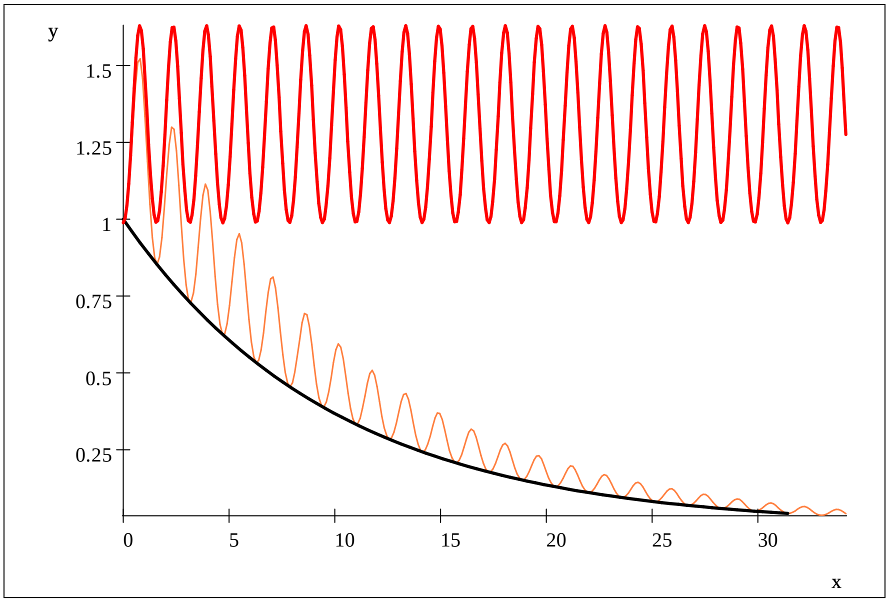
Introduce a "corrector" multiplicator $\Theta(t)$ introduced so that

$$\Phi(t, 0) = \Theta(t) \exp\left(t\frac{G}{p}\right)$$

Observe that

$$\Theta(t) = \frac{1}{0.778\,8} \exp(-0.25\cos(4.0t))$$

is a $p = 0.5\pi$ - periodic function equal to 1 in all points $t = pn = (0.5\pi)\, n$, $n = 1, 2, 3...$(red graf).



We are going to observe soon that a similar representation of the transition matrix $\Phi(t,0)$ is possible for the transition matrix $\Phi(t,0)$ of arbitrary periodic linear systems of ODEs.

The main idea of the Floquet theory.

The monodromy matrix $\Phi(p,0)$ is a particular transition matrix that maps initial data ξ at time $\tau = 0$ to the state of the system $x(p)$ after one period p. A particular property of this matrix in the case of periodic systems is that similar mapping to the state $x(t)$ at the time $t = np$ equal to n periods p is just

$$\Phi(n \cdot p, 0) = [\Phi(p,0)]^n$$

This property is similar to properties of autonomous linear systems where $\Phi(t,0) = \exp(At)$ *and therefore*

$$\Phi(n \cdot p, 0) = \exp(A(n \cdot p)) = [\exp(A(p))]^n = [\Phi(p,0)]^n \tag{12}$$

that follows from the factorisation property of the exponent of two commuting matrices:

$$\exp(A+B) = \exp(A)\exp(B)$$

In the case of periodic systems this factorisation applies only for shifts in time that are integer numbers of periods. But it is still a remarkable property. The behaviour of solutions is described by a repeated multiplication by a constant matrix in certain time points: p, $2p$, $3p$, ...:

$$x'(t) = A(t)x(t), \quad x(0) = \xi.$$

$$x(np) = [\Phi(p,0)]^n \xi, \quad n = 0, 1, 2, ...$$

The first idea of the Floquet theory is to represent $x(np)$ at times $t = np$ similarly as for autonomous systems, namely with the help of an exponent of some (unknown at the moment) constant matrix F times the time argument: $t = np$.

$$x(np) = [\Phi(p,0)]^n \xi = \exp(np\,F)\xi = [\exp(p\,F)]^n \xi$$

It means that the matrix F in such representation must satisfy the relation

$$\Phi(p,0) = \exp(pF).$$

Therefore the matrix pF must be something like the logarithm of the monodromy matrix:

$$pF = \log(\Phi(p,0))$$

Definition. A matrix $G \in \mathbb{C}^{N\times N}$ is called a **logarithm of the matrix** $H \in \mathbb{C}^{N\times N}$ if

$$H = \exp(G)$$

We write in this case $G = \log(H)$.

We are going to prove soon that for any non-singular matrix H there is a logarithml $\log(H)$ in this sense. Point out that the monodromy matrix $\Phi(p,0)$ is always non-singular, because columns in a transition matrix $\Phi(t,0)$ are always linearly independent (check yourself: why?)

The logarithm of a matrix is not uniquely defined in the same way as it is not unique for complex and real numbers z:

$$\ln(z) = \ln(|z|) + i\arg(z) \tag{13}$$

because the argument $\arg(z)$ of a complex number is defined only up to $2\pi k$, $k = \pm1, \pm2, ...$

One can choose a unique branch for the logarithm function, called the *principle logarithm* or $\text{Log}\,(z)$ by choosing the argument in the last formula (13) for example only in the interval $(-\pi, \pi]$.

We will suspend the discussion of matrix logarithm now and will consider first an application of it to the analysis of solutions to periodic linear systems of ODEs.

The main idea in the Floquet theory is the "approximation" of the transition matrix $\Phi(t,0)$ for a periodic linear system with matrix $A(t) = A(p+t)$ by the transition matrix $\exp(t\,F)$ for an autonomous system

$$y'(t) = [F]\,y(t)$$

with the constant matrix $F = \left[\frac{1}{p}G\right]$ where

$$\begin{aligned} G &= \log(\Phi(p,0)) && (14)\\ \exp(G) &= \Phi(p,0) && (15)\\ \exp(pF) &= \Phi(p,0) && (16)\\ \exp((np)F) &= [\Phi(p,0)]^n = \Phi(np,0) && (17) \end{aligned}$$

$\Phi(p,0)$_is_considered_as_a_transition_matrix_for_the _autonomous_system $y'(t) = [F]\,y(t)$.

These two transition matrices coinside in points $t = 0, p, 2p, 3p, ...$

$$\Phi(np,0) = [\Phi(p,0)]^n = \exp((np)\,[F]) \tag{18}$$

The "deviation" of $\Phi(t,0)$ from $\exp(t\,F)$ in intermediate points within one period can be expressed by a factor $\Theta(t)$ so that

$$\begin{aligned} \Phi(t,0) &= \Theta(t)\exp(tF)\\ \Theta(t) &= \Phi(t,0)\exp(-tF) \end{aligned}$$

The matrix function $\Theta(t)$ must be equal to the unit matrix I in the points $t = 0, p, 2p, ...$ because in these points these two transition functions coinside by construction and $\exp(-tF)$ is inversion of $\Phi(t,0)$, see (18).

The exact formulation of the properties of such factorization is given in the following Theorem by Floquet.

Theorem 2.30 , p. 53. Floquet theorem

Consider a periodic system $x'(t) = A(t)x(t)$, with period p: $A(t) = A(t+p)$

Let $G \in \mathbb{C}^{N\times N}$ be a logarithm of the monodromy matrix $\Phi(p,0)$.

$$G = \log(\Phi(p,0))$$

There exists a periodic with period p piecewise continuously differentiable function $\Theta(t) : \mathbb{R} \to \mathbb{C}^{N\times N}$, with $\Theta(0) = I$ and $\Theta(t)$ non-singular (invertible, all eigenvalues are non-zero) for all t, such that

$$\Phi(t,0) = \Theta(t)\exp\left(\frac{t}{p}G\right), \quad \forall t \in \mathbb{R} \tag{19}$$

Proof.

We remind the main property (8) of the monodromy matrix for $\tau = 0$:

$$\Phi(t+p,0) = \Phi(t+p,p)\Phi(p,0) = \Phi(t,0)\Phi(p,0)$$

where we applied first the Chapman Kolmogorov relation (5) and then the shift invariance (7) of the transition matrix function $\Phi(t,\tau)$ for a periodic linear system.

We denote $\frac{1}{p}G$ by F for convenience, so that $G = pF$, and define the matrix function $\Theta(t)$ after the desired relation (19)

$$\Theta(t) = \Phi(t,0)\exp\left(-\frac{t}{p}G\right) = \Phi(t,0)\exp(-tF)$$

The matrix function $\Theta(t)$ is well defined in such a way. The problem is to show that it has desired properties: p - periodicity and satisfies initial conditions.

We remind that $\Theta(0) = I$ and even $\Theta(np)= I$ for all $n = 0,1,2,3,...$because

$$\Phi(np,0) = [\Phi(p,0)]^n = \exp((np)F)$$

$\Phi(t,0)$ is piecewise contiuously differentiable or contiuously differentiable depending on if $A(t)$ is piecewise continuous or continuous. Therefore $\Theta(t)$ has the same property because $\exp\left(-\frac{t}{p}G\right)$ is continously differentiable. $\Theta(t)$ is also invertible for all t as a product of two invertible matrices $\Phi(t,0)$ and $\exp(-tF)$.

We check now that $\Theta(t)$ is p - periodic, namely that $\Theta(t+p) = \Theta(t)$ for all $t \in \mathbb{R}$.

$$\begin{aligned}
&\Theta(t+p) \stackrel{def}{=} \Phi(t+p,0)\exp(-(t+p)F) \\
=\ &\Phi(t+p,0)\exp(-pF)\exp(-tF) = \Phi(t+p,0)\overbrace{\exp(-G)}^{(\exp(G))^{-1}}\exp(-tF) \\
=\ &\Phi(t+p,0)\ \overbrace{(\exp(G))^{-1}}^{(\Phi(p,0))^{-1}=\Phi(0,p)}\ \exp(-tF)
\end{aligned}$$

We remind that $\exp(G) = \exp(\log(\Phi(p,0)) = \Phi(p,0)$, therefore, by Chapman-Kolmogorov relations $\exp(-G) = (\exp(G))^{-1} = \Phi(p,0)^{-1} = \Phi(0,p)$. Therefore, using the main relation for the monodromy matrix (**??**) $\Phi(t+p,0) = \Phi(t,0)\Phi(p,0)$ together with the relation $\exp(-G) = \Phi(0,p)$, we arrive to

$$\Theta(t+p) = \Phi(t,0)\overbrace{\Phi(p,0)\Phi(0,p)}^{\Phi(p,p)=I}\exp(-tF) = \Phi(t,0)\,(I)\exp(-tF) \stackrel{\text{def}}{=} \Theta(t),$$

where we also used that $\Phi(p,0)\Phi(0,p) = I$ in the last step. Therefore $\Theta(t)$ is periodic with period p.■

1.1 Logarithm of a matrix. Existence and calculation.

We will formulate a theorem and give a proof to it (simpler than in the book) about the existence of a matrix logarithm.

Definition

The matrix G is a **logarithm of matrix** H or $G = \log(H)$ if $\exp(G) = \exp(\log(H)) = H$.

Consider a nonsingular matrix H and it's a canonical Jordan form J:

$$H = TJT^{-1}$$

where T is invertible matrix. Then if there is $Q \in \mathbb{C}^{N\times N}$, such that $\exp(Q) = J$ it means that

$$Q = \log(J), \qquad J = \exp(Q)$$

then according to the properties of the exponent of similar matrices, and the definition of the matrix logarithm

$$\begin{aligned} H &= TJT^{-1} = T\exp(Q)T^{-1} = T\exp(\log(J))T^{-1} = \\ &= \exp\left(T\log(J)T^{-1}\right) \stackrel{\text{def}}{=} \exp(\log(H)) \end{aligned}$$

and

$$\log(H) = T\log(J)T^{-1}$$

where we used te relation for exponent for similar matrices: that if $A = TBT^{-1}$ then $\exp(A) = T\exp(B)T^{-1}$.

It means that to calculate the logarithm of an arbitrary matrix H it is enough to calculate the logarithm of it's Jordan canonical form. For $H = TJT^{-1}$

$$\log(H) = T\log(J)T^{-1}$$

Definition.

We say that G is a principal logarithm $G = \operatorname{Log}(H)$ of the matrix H if G is a matrix logarithm of H and

$$\begin{aligned} \sigma(H) &= \{\exp(\lambda): \quad \lambda \in \sigma(G)\} \\ \sigma(G) &= \{\operatorname{Log}(\mu): \quad \mu \in \sigma(H)\} \end{aligned}$$

where $\text{Log}(\mu)$ is the scalar principal logarithm:

$$z = e^{\text{Log}(z)}; \quad \arg(\text{Log}\,(z)) = \text{Im}(\text{Log}\,(z)) \in (-\pi, \pi].$$

This definition implies the explicit one to one correspondence between eigenvalues to H and eigenvalues to G. Essentially the second relation is non-trivial.

Theorem.Proposition 2.29, p. 53.

If $H \in \mathbb{C}^{N\times N}$ is invertible, then there exists a principle logarithm $\text{Log}\,(H)$.

Proof.

We have established above that it is enough to investigate existence of logarithm for the similar canonical Jordan form J of the matrix H. So without loss of generality we may assume that H is a canonical Jordan form J. Exponent of a Jordan matrix consists of exponents of it's blocks. Therefore it is enough to establish the existence of logarithm for each Jordan block J_j in J, $j = 1, ..., s$ where s is the number of distinct eigenvectors to H and J_j has size $n_j \times n_j$

$$J_j = \begin{bmatrix} \lambda_j & 1 & 0 & ... & 0 & 0 \\ 0 & \lambda_j & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & \lambda_j & 1 \\ 0 & 0 & 0 & ... & 0 & \lambda_j \end{bmatrix}$$

$J_j = \lambda_j \left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right)$ where

$$\mathcal{N}_j = \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 \end{bmatrix}$$

From the classical Maclaurin series for $\log(1+x) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} x^p$ valid for $|x| < 1$, and for exp we get

$$\exp(\log(1+x)) = 1 + x$$

We will try to calculate $\log(J_j) = \log\left(\lambda_j I \left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right)\right) = \log(\lambda_j) I + \log\left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right)$

Point out that we use the condition that $\lambda_j \neq 0$!!!

We formally write the Maclaurin series for $\log(I + \frac{1}{\lambda_j}\mathcal{N}_j)$:

$$\log\left(I + \frac{1}{\lambda_j}\mathcal{N}_j\right) = \sum_{p=1}^{n_j - 1} \frac{(-1)^{p+1}}{p} \left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p$$

and observe that the Maclaurin series for $\log(1+\frac{1}{\lambda_j}\mathcal{N}_j)$ is a **finite sum** because all larger powers of $\mathcal{N}_j$ in the series cancel. We have therefore that

$$\exp\left(\log\left(I+\frac{1}{\lambda_j}\mathcal{N}_j\right)\right) = I+\frac{1}{\lambda_j}\mathcal{N}_j$$

and

$$\begin{aligned} & \exp\left(\log(\lambda_j)I\right)\exp\left(\log\left(I+\frac{1}{\lambda_j}\mathcal{N}_j\right)\right) = \\ \exp\left(\log(\lambda_j)I+\log\left(I+\frac{1}{\lambda_j}\mathcal{N}_j\right)\right) &= \lambda_j\left(I+\frac{1}{\lambda_j}\mathcal{N}_j\right) = J_j \end{aligned}$$

We define

$$G_j \stackrel{\text{def}}{=} \log(\lambda_j)I + \sum_{p=1}^{n_j-1}\frac{(-1)^{p+1}}{p}\left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p$$

Now we check that this expression G_j is actually a matrix logarithm $G_j = \log(J_j)$ for the Jordan block J_j by just checking that is satisfies the definition of the matrix logarithm. Point out that the diagonal matrix $\log(\lambda_j)I$ commutes with any matrix. Therefore applying formula $\exp(\log(1+x)) = 1+x$ for series for $\exp(x)$ and $\log(1+x)$ to similar converging series of commuting matrices we arrive to the desired relation.

$$\begin{aligned} \exp(G_j) &= \exp\left(\log(\lambda_j)I + \sum_{p=1}^{n_j-1}\frac{(-1)^{p+1}}{p}\left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p\right) \\ &= \exp\left(\log(\lambda_j)I\right)\exp\left(\sum_{p=1}^{n_j-1}\frac{(-1)^{p+1}}{p}\left(\frac{1}{\lambda_j}\mathcal{N}_j\right)^p\right) \\ &= \exp\left(\log(\lambda_j)I\right)\exp\left(\log\left(I+\frac{1}{\lambda_j}\mathcal{N}_j\right)\right) = (\lambda_j I)\left(I+\frac{1}{\lambda_j}\mathcal{N}_j\right) = J_j \end{aligned}$$

In the Jordan canonical form J eigenvalues stand on diagonal and are easy to control. All calculations that we have carried out are correct because $\lambda_j \neq 0$. We can choose logarithms $\log(\lambda_j)$ in these calculations as principle values of logarithm $\mathrm{Log}(\lambda_j)$. In this case the logarithm of J_j will be principal logarith, because there will be one to one correspondence between eigenvalues λ_j to J_j and eigenvalues $\mathrm{Log}\,(\lambda_j)$ to $\mathrm{Log}\,(J_j)$ that are diagonal elements in corresponding matrices. They will have the same algebraic multiplicity and the same geometric multiplicity 1 (one linearly independent eigenvector for each Jordan block)

Therefore the existense of the principal logarithm is established also for J and for H, that is a matrix similar to J. The same correspondence as above is valid for the eigenvalues to H and to $\mathrm{Log}(H)$ because eigenvalues to similar matrices H and J are the same. The number of linearly independent eigenvectors corresponding to each distinct eigenvalue (geometric multiplicity) will be also the same.■

1.2 Floquet multiplicators and exponents and bounds of solutions to periodic systems. equations.

Definition.

Eigenvalues of the monodromy matrix $\Phi(p,0)$ are called **Floquet's multipliers** or **characteristic mutipliers.**

A Floquet multiplier is called semisimple if it is semisimple as an eigenvalue to the monodromy matrix $\Phi(p,0)$.

Definition.

Eigenvalues of the logarithm of the monodromy matrix are called **Floquet's exponents or characteristic exponents.**

Theorem 2.31, p.54 on boundedness and zero limits of solutions to periodic linear systems.

1) Every solution to a periodic linear system of ODEs is bounded on $\mathbb{R}_+$ if and only if the abosolute value of each Floquet multiplier is not greater than 1 and any Floquet multiplier with absolute value 1 is semisimple.

2) Every solution to a periodic linear system of ODEs tends to zero at $t \to \infty$ if and only if the absolute value of each Floquet multiplier is strictly less than 1.

Proof. (required at the exam)

By the Floquet theorem any solution $x(t)$ to system

$$x'(t) = A(t)x(t), \quad A(t+p) = A(t), \forall t \in \mathbb{R} \tag{20}$$

satisfying initial conditions

$$x(\tau) = \xi$$

is represented as

$$\begin{aligned} x(t) &= \Phi(t,\tau)\xi = \overbrace{\Theta(t)\exp(tF)}^{\Phi(t,0)}\Phi(0,\tau)\xi = \Theta(t)\overbrace{\exp(tF)\zeta}^{y(t)} \\ &= \Theta(t)y(t) \end{aligned}$$

where

$$F = \frac{1}{p}\mathrm{Log}(\overbrace{\Phi(p,0)}^{monodromy\_matrix}), \quad \zeta = \Phi(0,\tau)\xi.$$

$\Theta(t)$ is a p - periodic continuous or piecewise continuous matrix valued function. $\Theta(t)$ is invertible for all t.

We define $y(t) = \exp(tF)\zeta$ as a solution to the I.V:P. for an autonomous linear equation:

$$y'(t) = F\,y, \quad y(0) = \zeta \tag{21}$$

$y(t) = \Theta^{-1}(t)x(t)$, and $x(t) = \Theta(t)y(t)$. The mapping $\Theta(t)$ determines a one to one correspondence between solutions $x(t)$ to the periodic system (20) and solutions $y(t)$ to the autonomous system (21).The periodicity and continuity properties of $\Theta(t)$ and $\Theta^{-1}(t)$ imply that there is a constant $M > 0$ such that $\|\Theta(t)\| \leq M$ and $\|\Theta^{-1}(t)\| \leq M$ for all $t \in \mathbb{R}$. It implies that

$$\|x(t)\| \leq M \|y(t)\| \quad and \quad \|y(t)\| \leq M \|x(t)\|$$

Therefore

1)$\|x(t)\|$ is bounded on $\mathbb{R}_+$ if and only if corresponding $\|y(t)\| = \|\exp(tF)\zeta\|$ is bounded on $\mathbb{R}_+$.

2) $\|x(t)\| \to 0$ when $t \to \infty$ if and only if corresponding $\|y(t)\| \to 0$ when $t \to \infty$.

Since $\text{Log}\,(\Phi(p,0)) = G = pF$, and $\Phi(p,0) = \exp(pF)$ it follows that

$$\begin{aligned} \sigma\,(\Phi(p,0)) &= \{\exp(\lambda p): \quad \lambda \in \sigma(F)\} \\ \sigma\,(F) &= \left\{\frac{1}{p}\text{Log}(\mu) : \mu \in \sigma(\Phi(p,0))\right\} \end{aligned}$$

and that algebraic and geometric multiplicities of each $\lambda \in \sigma(F)$ coinside with those of $\exp(p\lambda) \in \sigma\,(\Phi(p,0))$.

The following connections between the properties of Floquet multipliers and propertis of corresponding eigenvalues to the matrix $F = \frac{1}{p}\text{Log}(\Phi(p,0))$ are direct consequences of the Euler formula for the complex exponent and properties of complex logarithm:

$$\begin{aligned} \text{Log}(z) &= \ln(|z|) + i\text{Arg}(z) \\ \exp(z) &= \exp(\text{Re}\, z)(\cos(\arg z) + i\sin(\arg z)) \\ |\exp(z)| &= \exp(\text{Re}\, z) < 1 \Longleftrightarrow \text{Re}\, z < 0 \\ |\exp(z)| &= \exp(\text{Re}\, z) \leq 1 \Longleftrightarrow \text{Re}\, z \leq 0 \\ |\exp(z)| &= \exp(\text{Re}\, z) = 1 \Longleftrightarrow \text{Re}\, z = 0 \end{aligned}$$

a) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$,has$|\mu| < 1$ if and only if $\text{Re}\,\text{Log}(\mu) < 0$ that is if the corresponding eigenvalue $\lambda = \frac{1}{p}\text{Log}(\mu)$ to F has $\text{Re}\,\text{Log}(\mu) < 0$.

b) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$,has $|\mu| \leq 1$ if and only if $\text{Re}\,\text{Log}(\mu) \leq 0$ that is if the corresponding eigenvalue $\lambda = \frac{1}{p}\text{Log}(\mu)$ to F has $\text{Re}\,\text{Log}(\mu) \leq 0$.

c) The Floquet multiplier $\mu \in \sigma(\Phi(p,0))$, with $|\mu| = 1$ is semisimple if and only if the corresponding eigenvalue $\lambda = \frac{1}{p}\text{Log}(\mu)$ to F having $\text{Re}\,\text{Log}(\mu) = 0$ is semisimple.

Known relations between properties of solutions to an autonomous system and the spectrum of corresponding matrix applied to the system $y'(t) = F\,y$ and to the spectrum $\sigma(F)$ of the matrix F together with statements 1),2), a),b),c) in the present proof imply the statement of the theorem.■

Proposition 2.20. p. 45. On periodic solutions to periodic linear systems

The system $x'(t) = A(t)x(t)$ with p - periodic $A(t) = A(t+p)$ has a non-zero p - periodic solution if and only if the monodromy matrix $\Phi(p,0)$ has an eigenvalue $\lambda = 1$. A more general statement is also valid.

The system has a non-zero $n\,p$ - periodic solution for $n \in \mathbb{N}$ if and only if the monodromy matrix $\Phi(p,0)$ has an eigenvalue λ such that $\lambda^n = 1.\square$

Proof. Consider an eigenvector v corresponding to the eigenvalue λ. Then $v \neq 0$, $\Phi(p,0)v = \lambda v$ and

$$[\Phi(p,0)]^n\, v = \lambda^n v = 1 \cdot v = v$$

We will show that the solution to the system, with initial data $x(0) = v$ has period np. This solution is given by the transition matrix: $x(t) = \Phi(t,0)v$. Using this representation and applying the factorisation property of transition matrices for periodic systems we arrive to

$$x(t+np) = \Phi(t+np,0)v = \Phi(t,0)\,[\Phi(p,0)]^n\, v = \Phi(t,0)v = x(t), \quad \forall t \in \mathbb{R}$$

It shows that $x(t)$ is periodic with period $n\,p$.

Supposing that there is a periodic solution $x(t+np) = x(t)$ and repeating the same calculation backwards we arrive to that $x(0) = v$ is an eigenvector corresponding to an eigenvalue λ such that $\lambda^n = 1$.

Carry out this backward argument as an exercise!

Examples and exercises on periodic and general linear systems of ODEs.

Exercise. Compute the monodromy matrix for the system $x'(t) = A(t)x(t)$ with the following periodic matrix $A(t)$ with period 1 and find conditions on α that imply that all solutions tend to zero with $t \to \infty$.

$$A(t) = \begin{cases} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} = A_1, & 0 \le t < 1/2 \\ \\ \begin{bmatrix} \alpha & 0 \\ 1 & \alpha \end{bmatrix} = A_2, & 1/2 \le t < 1 \end{cases}$$

Solution:

The monodromy matrix $\Phi(p,0) = \Phi(1,0)$ is expressed as (using Chapman- Kolmogorov)

$$\begin{aligned} \Phi(1,0) &= \Phi(1,1/2)\Phi(1/2,0) \\ &= \exp((1-1/2)A_2)\exp((1/2)\,A_1) \\ &\quad \exp((1/2)A_2)\exp((1/2)\,A_1) \end{aligned}$$

Here $\exp(tA_1) = \exp(\alpha t)\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, $\exp(tA_2) = \exp(\alpha t)\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$

We derive an explicit expression for $\Phi(1,0)$ $\Phi(1,0) = \exp(\frac{1}{2}\alpha + \frac{1}{2}\alpha)\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} = \exp(\alpha)\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix}$,

$\det \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix} = 1$; $Tr \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{bmatrix} = 2.25.$

characteristic polynomial $p(\lambda) = \lambda^2 - \frac{9}{4}\lambda + 1$

eigenvalues: $\lambda_1 = \frac{9}{8} - \sqrt{\left(\frac{9}{8}\right)^2 - 1} = \frac{9}{8} - \frac{1}{8}\sqrt{17} > 0$, $\lambda_2 = \frac{1}{8}\sqrt{17} + \frac{9}{8} > 0$ and are simple.

Find conditions on α such that all solutions will be bounded

The condition for boundedness of all solutions is $\exp(\alpha)\,|\lambda_2| \leq 1$ or $\exp(\alpha)\frac{1}{8}\left(\sqrt{17}+9\right) \leq 1$ because λ_2 is larger in absolute value.

It can be reformulated by taking logarithm of left and right hand sides as $\alpha \leq \ln(8) - \ln(\sqrt{17}+9) \approx -0.494\,93$.

All solutions will tend to zero if and only if the strict inequality is valid $\alpha < \ln(8) - \ln(\sqrt{17}+9) \approx -0.494\,93$

■

Abel - Liouville's formula

Consider a homogeneous linear system of ODEs $x'(t) = A(t)x(t)$ and N solutions $y_1(t), y_2(t),...,y_N(t)$ to it. Consider the matrix $Y(t)$ having these solutions as it's columns:

$$Y(t) = [y_1(t), y_2(t), ..., y_N(t)]$$

Definition.

The determinant

$$w(t) = \det Y(t) = \det\,[y_1(t), y_2(t), ..., y_N(t)]$$

is called **Wronskian** associated with solutions $y_1(t), y_2(t), ..., y_N(t)$.

Proposition 2.7 part (2) - Abel - Liouville's formula

Wronskian $w(t)$ associated with solutions $y_1(t), y_2(t),...,y_N(t)$ to the system $x'(t) = A(t)x(t)$ satisfies the following relations:

$$w(t) = \det \Phi(t,\tau) w(\tau)$$

In points t where $A(t)$ is continuous it satisfies the differential equation

$$w'(t) = \mathrm{tr}(A(t))w(t)$$

and therefore with initial value for $w(\tau)$ at time τ :

$$w(t) = w(\tau)\exp\left(\int_\tau^t \mathrm{tr}(A(s))ds\right) \tag{22}$$

for all $t \in J$.■

Corollary 2.33, p. 59

We consider a periodic linear system $x'(t) = A(t)x(t)$, $A(t+p) = A(t)$.

If $\int_0^p \mathrm{tr}(A(s)ds$ has a positive real part, then the equation has at least one solution $x(t)$ that is unbounded, or expressing it more explicitly, the upper limit of it's norm is infinite: $\limsup_{t\to\infty} \|x(t)\| = \infty$.

□

Proof.

We remind that the transition matrix $\Phi(t,\tau)$ satifies the initial value problem:

$$\begin{aligned} \frac{d}{dt}\Phi(t,\tau) &= A(t)\Phi(t,\tau) \\ \Phi(\tau,\tau) &= I \end{aligned}$$

Arbitrary solution to the initial problem $x'(t) = A(t)x(t)$, $x(\tau) = \xi$ will be expressed as

$$x(t) = \Phi(t,\tau)\xi$$

According to Abel - Liouville's formula and considerations before

$$\det(w(t,0)\,) = \det(w(0,0)) \exp\left(\int_0^t tr(A(s)ds\right)$$

It implies for $\Phi(t,0)$

$$\begin{aligned} |\det(\Phi(t,0)\,)| &= \left|\det(\Phi(\overset{=I}{0,0})) \exp\left(\int_0^t tr(A(s)ds\right)\right| = \\ \left|\exp\left(\int_0^t tr(A(s)ds\right)\right| &= \left|\exp\left(\mathrm{Re}\left(\int_0^t tr(A(s)ds\right)\right)\right| \end{aligned}$$

Therefore, for $t = p$ we get that if $\mathrm{Re}\left(\int_0^p tr(A(s)ds\right) > 0$ then

$$|\det(\Phi(p,0)\,)| = \left|\exp\left(\mathrm{Re}\int_0^p tr(A(s)ds\right)\right| > 1.$$

On the other hand $\det(\Phi(p,0))$ is a product of eigenvalues μ_k to the monodromy matrix $\Phi(p,0)$ with multiplicities m_k

It follows from the structure of the similar Jordan matrix

$$\begin{aligned} \Phi(p,0) &= T^{-1}JT \\ \det\Phi(p,0) &= \det(T)^{-1}\det(T)\det(J) = \det(J) \end{aligned}$$

$$|\det(\Phi(p,0)| = \prod_{k=1}^{s} |\mu_k|^{m_k} > 1$$

To have this product greater than 1 we must have at least one eigenvalue μ_p with $|\mu_p|$ >1. According to one of Floquet theorems, all solutions to a periodic system of ODEs are bounded if and only if all eigenvalues to the monodromy matrix have absolute value $|\lambda| \leq 1$ and those with $|\lambda| = 1$ are semisimple. Therefore existence of an eigenvalue with $|\lambda| > 1$ implies existence of a solution $x(t)$ that is not bounded and therefore $\limsup_{t\to\infty} \|x(t)\| = \infty$. ■

For example we can choose the initial condition $x(0) = v_p$ with v_p being the eigenvector to $\Phi(p, 0)$ corresponding to the eigenvalue $|\mu_p| > 1$.Then the solution

$$\begin{aligned} x(t) &= \Phi(t, 0)v_p \\ Let\_t_n &= np \\ \Phi(np, 0)v_p &= [\Phi(p, 0)]^n v_p = (\mu_p)^n v_p \end{aligned}$$

with $|\mu_p| > 1$. Therefore $x(t_n) = (\mu_p)^n v_p$, and

$$\|x(t_n)\| = |\mu_p|^n \|v_p\| \underset{n\to\infty}{\to} \infty$$

and $\limsup_{t\to\infty} \|x(t)\| = \infty$.

□

We give also a geometric interpretation of this result. Consider a unite cube build on standard base vectors $e_1, ..., e_N$ at time $t = 0$. Consider how the volume $\mathrm{Vol}(t)$ of this cube changes under the action of the linear transformation by the transition matrix $\Phi(t, 0)$ of our periodic system. Point out that $I = [e_1, ..., e_N]$ .It implies that the figure of interest is the parallelepiped build on columns of the transition matrix $\Phi(t, 0)$. One of the main properties of periodic system is that $\Phi(np, 0) = [\Phi(p, 0)]^n$. Therefore

$$\mathrm{Vol}(np) = |\det([\Phi(p, 0)]^n)| = |\det([\Phi(p, 0)])|^n = \left[\exp\left(\mathrm{Re}\left(\int_0^p tr(A(s)ds)\right)\right)\right]^n$$

If $\mathrm{Re}\left(\int_0^p tr(A(s)ds)\right) > 0$ then $\exp\left(\mathrm{Re}\left(\int_0^p tr(A(s)ds)\right)\right) > 1$. It implies that

$$\lim_{n\to\infty} \mathrm{Vol}(n\,p) = \infty$$

Therefore along the sequence of times$\{t = np, \quad n = 1, 2, 3, ...\}$ we obsere that $\mathrm{Vol}(n\,p)$ is unbounded. It implies also that

$$\limsup_{t\to\infty} \|\mathrm{Vol}(t)\| = \infty$$

The fact that $\lim_{n\to\infty} \mathrm{Vol}(n\,p) = \infty$ implies that the diameter $D(n\,p)$ of the parallelepiped build on columns of $\Phi(n\,p, 0)$ calculated at these discrete time points, is also unbounded $\lim_{n\to\infty} D(n\,p) = \infty$. It in turn means that there should be a solution that has the property $\lim_{n\to\infty} \|x(np)\| = \infty$.□