1 Banach's contraction principle. Picard-Lindelöf theorem.

We consider in this chapter the theorem by Picard and Lindelöf about existence and uniqueness of solutions to the initial value problem to the system of differential equations in the form

$$x'(t) = f\,(t, x(t)) \tag{1}$$

$$x(\tau) = \xi \tag{2}$$

Here $f : J \times G \to \mathbb{R}^n$ is a vector valued function continuous with respect to time variable t and space variable x. J is an interval, G is an open subset of $\mathbb{R}^n$.

One can reformulate the I.V.P. (1),(2) in the form of the integral equation

$$x(t) = \xi + \int_{\tau}^{t} f\,(s, x(s))\,ds \tag{3}$$

If f is continuous, then these two formulations are equivalent by the Newton-Leibnitz theorem.

If f is only peacewise continuous in time t, then these formulations are equialent on intervals of continuity of f in time and solutions can be glued by continuity of the solution in the points were the derivative in time doses not exist.

Fixed points of operators.

Consider a vector space X with a subset $C \subset X$ and an operator $K : C \to C$.

Definition

A point $\overline{x} \in C$ is called the **fixed point** of the operator K on the set C if

$$K(\overline{x}) = \overline{x} \tag{4}$$

A general idea behind the analysis of many types of non-linear equations is to reformulate them as a fixed point problem.

Consider the right hand side of the integral equation (3) corresponding to the I.V.P as an operator

$$K(x)(t) \stackrel{\text{def}}{=} \xi + \int_{\tau}^{t} f\,(s, x(s))\,ds$$

acting from the vector space of vector valued continuous functions $C(I)$, where $I \subset J$ is a closed interval including τ . Point out that t can be smaller than τ $(t < \tau)$.

The expression $\|x\|_{C(I)} = \sup_{t \in I} \|x(t)\|$ defines a norm on the space $C(I)$ because it satisfies the triangle inequality and we know that uniformly convergent sequences of continuous functions on the compact set (I in this case) converge to continuous functions.

This space is even complete. It means per definition that Cauchy sequences of functions in $C(I)$ converge uniformly to continuous functions. It means more explicitely that if the sequence $\{x_n\} \in C(I)$ has the Cauchy property:

$$\|x_m - x_n\|_{C(I)} = \sup_{t \in I} \|x_m(t) - x_n(t)\|_{C(I)} \underset{m,n \to \infty}{\to} 0$$

then there is a continuous function $\overline{x} \in C(I)$ such that $x_n \underset{n \to \infty}{\to} \overline{x}$ uniformly on I, or what is the same, $\|x_n - \overline{x}\|_{C(I)} \underset{n \to \infty}{\to} 0$.

Definition.

We call a normed vector space a Banach space if it is complete with respect to it's norm.

This notion was introduced by Polish mathematician Stefan Banach who lead the greatest school in functional analysis at the university of Lwiv in Poland in the first half of the 20th century.

Examples.

1) The space $C(I)$ is a Banach space.

2) Elementary examples of Banach spaces are given by $\mathbb{R}^n$ supplied with norms $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ with $p \geq 1$.

3) A slight extension of this example is a set l_p, $p \geq 1$ of real sequences $\{x_i\}_{i=1}^{\infty}$ with finite norms in the form $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$.

4) One of the most popular classes of Banach spaces is the space of "integrable functions" $f : G \to \mathbb{R}$ where $G \subset \mathbb{R}^n$, with norms $\|f\|_{Lp} = \left(\int_G |f(z)|^p \, dz\right)^{1/p}$

"Integrable functions" and the integral here are in the sense of Lebesque, that is a contemporary notion of integral, studied in the course "Integration theory" given for master and for PhD students.

Remark.

We point out for convenience that different norms are used through out the text. Notation $\|\|$ means usual euclidean norm in $\mathbb{R}^n$. For a Banach space X the notation $\|x\|_X$ means the norm in the space X.

The operator K defined above, acts from $C(I)$ to itself. It makes that the I.V.P. above can be considered as a fixed value problem (**4**) on the whole $C(I)$ or on some subset of it.

A classical theorem that guarantees the existence and uniqueness of fixed points to non-linear operators in Banach and more generally in metric spaces, is Banach's contraction principle.

Definition. Operator $K : A \to A$, where $A \subset X$, and X is a Banach space, is called **contraction** on A if there is a constant $0 < \theta < 1$ such that for any $x, y \in A$

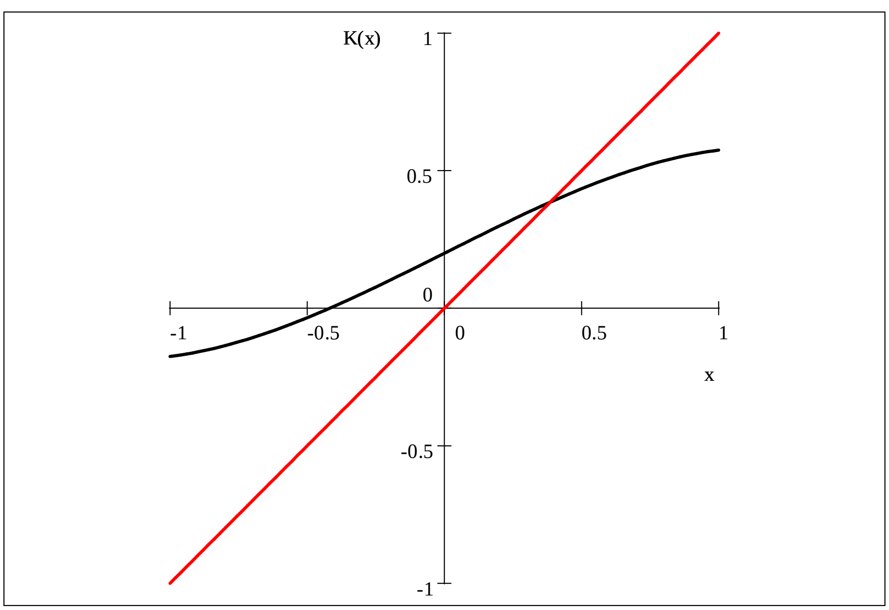
$$\|K(x) - K(y)\|_X \leq \theta \, \|x - y\|_X$$

Example. An elementary example is a smooth (at least C^1) function K acting from an interval $[a, b]$ to itself and having absolute value of the derivative $\left|\frac{d}{dt}K(t)\right| < \theta < 1$ for all $t \in [a, b]$. By intermediate value theorem for any $x, y \in [a, b]$ there is a point $c \in (x, y)$ such that $K(x) - K(y) = (x - y)K'(c)$.Therefore

$$|K(x) - K(y)| = |(x - y)| \, |K'(c)| \leq \theta \, |(x - y)|$$

It implies that K is a contraction in on the interval $[a, b]$.

Example: $K(x) = 0.5\,(x - 0.25x^3) + 0.2$ on $[-1, 1]$



Another example could be a Lipschitz function with Lipschitz constant L strictly smaller then one: $L < 1$.

Banach's contraction principle.

Theorem

Banach's contraction principle.

Let A be a non-empty closed subset of a Banach space X and $K : A \to A$ be a contraction operator with contraction constant $\theta < 1$ (strictly smaller than 1!)

Then there is a unique fixed point $\overline{x}$ to K in A such that $K\overline{x} = \overline{x}$.

More over, if $K^n(x_0) \stackrel{def}{=} K(K(...K(x_0))...)$ is the operator K applied to itself n times then for arbitrary initial approximation $x_0 \in A$, successive approximations $K^n(x_0)$ satisfy the estimate

$$\|K^n(x_0) - \overline{x}\|_X \leq \frac{\theta^n}{1-\theta} \|K(x_0) - \overline{x}\|_X$$

Proof is based on showing that the sequence of approximations $\{x_n\}_{n=0}^{\infty}$ defined by the equations

$$\begin{aligned} x_1 &= K(x_0) \\ &... \\ x_{n+1} &= K(x_n) \end{aligned}$$

with an arbitrary initial approximation $x_0 \in A$, converge to some $\overline{x} \in A$ that is the unique fixed point of K in A.

It follows by induction that

$$\begin{aligned} \|x_{n+1} - x_n\|_X &= \|K(x_n) - K(x_{n-1})\|_X \leq \theta \|x_n - x_{n-1}\|_X \\ &\leq \theta \|K(x_{n-1}) - K(x_{n-2})\|_X \leq \theta^2 \|x_{n-1} - x_{n-2}\|_X \\ &... \\ &\leq \theta^n \|x_1 - x_0\|_X \end{aligned}$$

We will show that $\{x_n\}_{n=0}^{\infty}$ is a Chauchy sequence using telescoping sequences. Let

$m > n$.

$$
\begin{aligned}
\|x_m - x_n\|_X &= \\
&= \|x_m - x_{m-1} + x_{m-1} - x_{m-2} + ... + x_{n+1} - x_n\|_X \\
&\overset{triangle\_inequality}{\leq} \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + ... + \|x_{n+1} - x_n\| \\
&\leq \left(\theta^n + \theta^{n-1} + ... + \theta^{m-1}\right) \|x_1 - x_0\|_X \\
&= \theta^n \left(1 + \theta + ...\theta^{m-n-1}\right) \|x_1 - x_0\|_X \\
&\leq \theta^n \underbrace{\left(1 + \theta + ...\theta^{m-n-1} + ...\right)}_{=\frac{1}{1-\theta}} \|x_1 - x_0\|_X \\
&\leq \theta^n \left(\frac{1}{1-\theta}\right) \|x_1 - x_0\|_X \to 0, \qquad n \to \infty, \quad \theta < 1
\end{aligned}
$$

The Banach space X is complete therefore the limit $\lim_{n\to\infty} x_n = \overline{x}$ exists. The set A is closed, therefore $\overline{x} \in A$.

Claim: $\overline{x}$ is a fixed point to K.

It is a non-trivial step in many approximation methods to show that an existing limit of approximations is a solution to the non-linear equation of interest. Here the convergence is strong, that makes the proof of the clime straightforward.

We see it by tending to the limit in the expression for x_n:

$$x_{n+1} = K(x_n)$$

$$
\begin{aligned}
\lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} K(x_n) = K(\lim_{n\infty} x_n) \\
\overline{x} &= K(\overline{x})
\end{aligned}
$$

and using the continuity of K.

The last question we must answer is the uniqueness of the fixed point to K in A.

Suppose that there is another fixed point $\widetilde{x}$ to K in A. Consider the norm of the difference $\overline{x} - \widetilde{x}$:

$$\|\overline{x} - \widetilde{x}\|_X = \|K(\overline{x}) - K(\widetilde{x})\|_X \leq \theta \|\overline{x} - \widetilde{x}\|_X \ , \quad \theta < 1$$

It is possible only if $\overline{x} - \widetilde{x} = 0$.

Finally we prove the estimate of the error in the approximations.

$$
\begin{aligned}
\|x_m - x_n\|_X &\le \theta^n \left(\frac{1}{1-\theta}\right) \|x_1 - x_0\|_X \\
\lim_{m\to\infty} \|x_m - x_n\|_X &\le \theta^n \left(\frac{1}{1-\theta}\right) \|x_1 - x_0\|_X \\
& \text{norm\_is\_a\_continuous\_function} \\
\left\| \lim_{m\to\infty} x_m - x_n \right\|_X &\le \theta^n \left(\frac{1}{1-\theta}\right) \|x_1 - x_0\|_X \\
\|\overline{x} - x_n\|_X &\le \theta^n \left(\frac{1}{1-\theta}\right) \|x_1 - x_0\|_X
\end{aligned}
$$

■

Elementary exercises on Banach's contraction principle.

Show using Banach's contraction principle that the polynomial $K(x) = x^2 - 0.4$ has a fixed point $K(x) = x$.

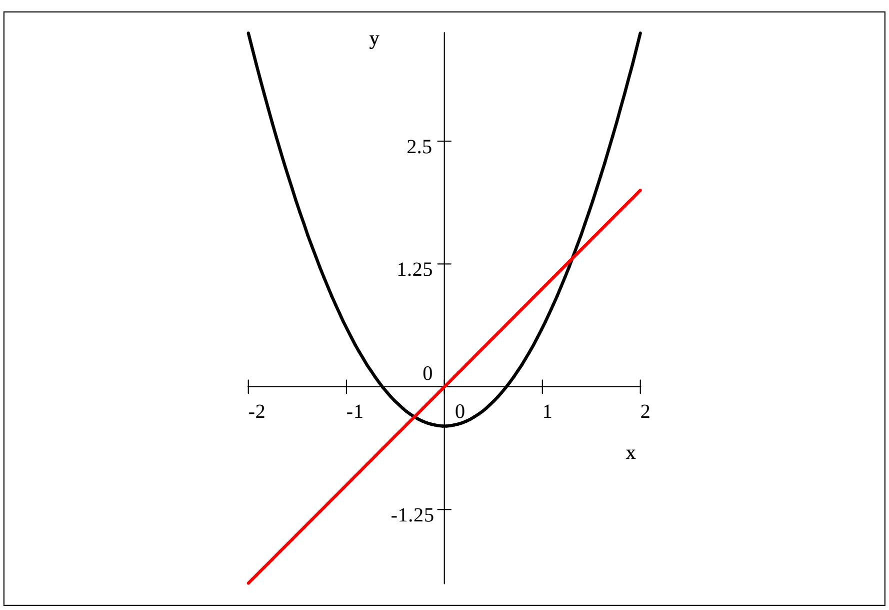
Solution consists of two steps.

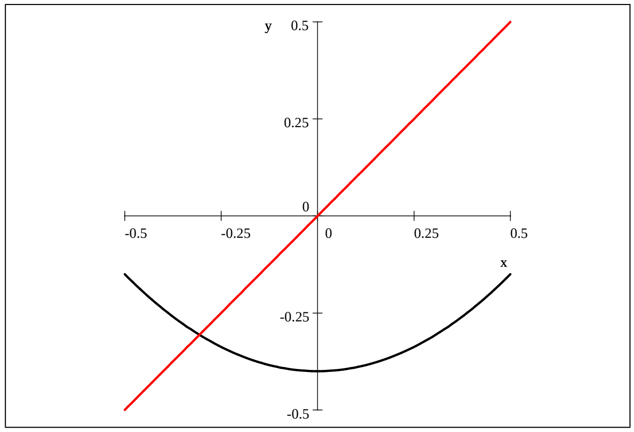
i) Find a set $B \subset \mathbb{R}$ where $K(x)$ has the contraction property: $|K(x) - K(y)| \le \theta |x - y|$, $\theta < 1$, for $x, y \in B$

ii) Find a subset $A \subset B$ that the function K maps into itself: $K : A \to A$.

i) $K'(x) = 2x < 1 \Longrightarrow x \in [-0.5 + \delta, 0.5 - \delta]$

ii) The set $[-0.5 + \delta, 0.5 - \delta]$ satisfies the requirement.




y
0.5
0.25
0
-0.5
-0.25
0
0.25
0.5
x
-0.25
-0.5

Picard-Lindelöf theorem.

Picard-Lindelöf theorem.

Here $f : J \times G \to \mathbb{R}^n$ is a vector valued function continuous in $J \times G$. J is an interval, G is an open subset of $\mathbb{R}^n$. Let in addition suppose that f is Lipschitz continuous with respect to the second argument with the Lipschitz constant $L > 0$:

$$\|f(t,x) - f(t,y)\| \leq L \|x - y\|, \forall x, y \in G$$

(We could suppose a weaker condition that this Lispchitz property is only local, but will not do it because it would make the proof just slightly longer without changing main ideas).

Then for any $(\tau, \xi) \in J \times G$ the initial value problem

$$\begin{aligned} x' &= f(x,t) \\ x(\tau) &= \xi \end{aligned}$$

has a unique solution on some time interval including τ. □

Remark. This local solution can always be extended to a unique maximal solution. We considered maximal extensions earlier in the course.

Proof to the Picard-Lindelöf theorem.

The proof is based on using the integral form of the I.V.P.

$$x(t) = \xi + \int_\tau^t f(s, x(s))\, ds$$

and applying Banach's contraction principle to it. We use the Banach space of continuous functions $x : I \to \mathbb{R}^n$on some compact interval $I \subset J$.

The application of Banach's principle here consists of two steps.

• The first step is to find a time interval I_1 and a closed subset $A \subset C(I_1)$ such that the operator K defined by

$$K(x)(t) = \xi + \int_\tau^t f(s, x(s))\, ds$$

maps A to itself: $K : A \to A$.

• The second step is to find a time interval I_2 such that the contraction property for the operator would be valid on a subset of $C(I_2)$. Finally we will choose the smallest of I_1 and I_2 for both properties to be valid and will conclude the result.

We consider here first the case with t on some interval $[\tau, \tau+T_{first}] \in J$, $T_{first} > 0$ and try to find a solution on this time interval (actually on some shorter time interval $[\tau, \tau+T]$ with $T < T_{first}$). Considering a time interval backword direction in time is similar

We choose first a closed ball $\overline{B(\xi,\delta)} = \{x : \|x-\xi\| \leq \delta\}$ such that it belongs to G: $\overline{B(\xi,\delta)} \in G$.

Our intension is to find solution in the set of continuous functions $x : [\tau, \tau+T] \to \mathbb{R}^n$ such that the solution $x(t) = \varphi(t,\tau,\xi) \in \overline{B(\xi,\delta)}$ for all $t \in [\tau, \tau+T]$ and therefore $\sup_{t\in[\tau,\tau+T]} \|x(t)-\xi\| \leq \delta$. It is a closed ball

$$A = \|x-\xi\|_{C([\tau,\tau+T])} = \sup_{t\in[\tau,\tau+T]} \|x(t)-\xi\| \leq \delta$$

in the infinitely dimensional space $C([\tau, \tau+T])$.

Our goal in the proof is to find such an interval $[\tau, \tau+T]$ that this set A in $C([\tau, \tau+T])$ and the operator K satisfy conditions in the Banach contraction principle.

The function $f(t,x)$ is continuous on the compact set in $V = [\tau, \tau+T_{first}] \times \overline{B(\xi,\delta)}$ that is a cylinder in $\mathbb{R}^{n+1}$, and therefore

$$M = \sup_{(t,x)\in V} \|f(t,x)\| < \infty$$

Point out that here we still operate on the "large" initial time interal $[\tau, \tau+T_{first}]$.

The constant M controls how large is the velocity $f(t,x)$ inside the set $V = [\tau, \tau+T_{first}] \times \overline{B(\xi,\delta)}$(yellow in the picture). Correspondingly M controls how fast the (blue) trajectory $x(t) = \varphi(t,\tau,\xi)$ can go away from the initial point ξ.

According to the integral equation for x

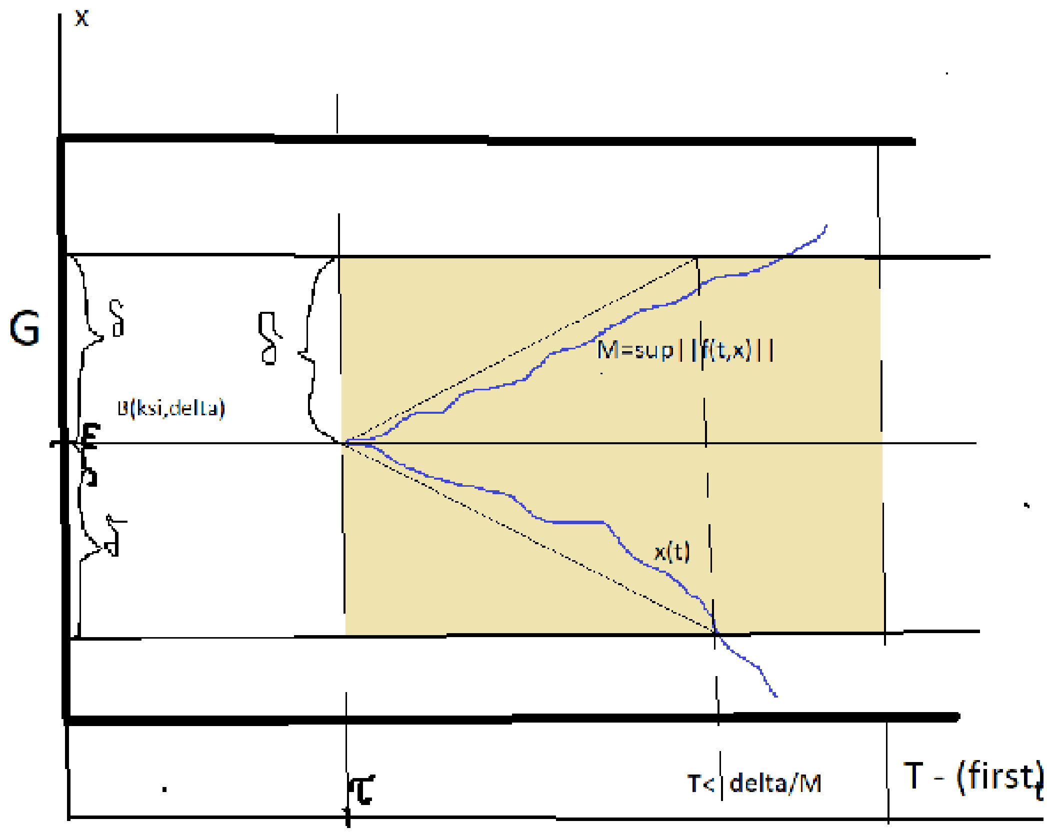
$$x(t) = \xi + \int_\tau^t f(s, x(s))\, ds$$

and by the estimate for f above, $x(t)$ must be inside the "angle" bounded by the cone $\|x-\xi\| = M(t-\tau)$ in $\mathbb{R}^{n+1}$.

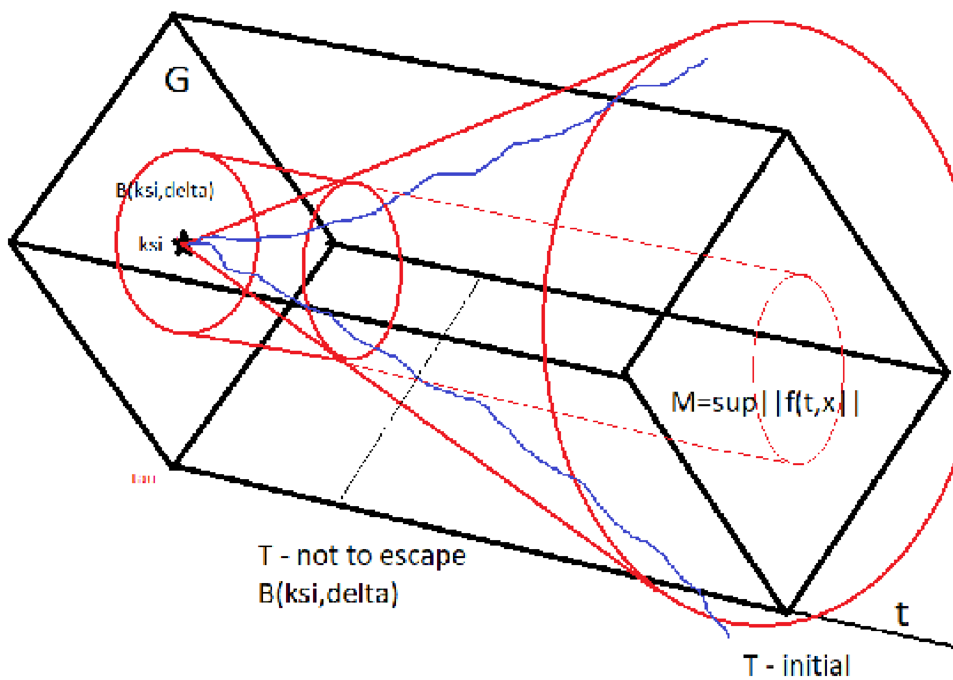
We give here two pictures illustrating the proof, a one dimensional picture:

and a two-dimensional picture:

We are going to estimate $\|K(x)(t)-\xi\|$ and choose the length T of the time interval $[\tau, \tau+T]$ in such a way that for any $x(t) \in \overline{B(\xi,\delta)}$ for $t \in [\tau, \tau+T]$, it follows that



x
G
δ
δ
B(ksi,delta)
ξ
δ
M=sup||f(t,x)||
x(t)
τ
T<delta/M
T - (first)
t


G
B(ksi,delta)
ksi
M=sup||f(t,x)||
T - not to escape
B(ksi,delta)
t
T - initial

$K(x(t))$ does not escape the ball $\overline{B(\xi,\delta)}$ around ξ in G.

$$\|K(x(t))-\xi\| \leq \delta$$

for $t \in [\tau, \tau+T]$.

It would imply after taking the supremum over $t \in [\tau, \tau+T]$ that

$$\sup_{t\in[\tau,\tau+T]} \|K(x)(t)-\xi\| = \|K(x)-\xi\|_{C([\tau,\tau+T])} \leq \delta$$

for $\|x-\xi\|_{C([\tau,\tau+T])} \leq \delta$. **Here we do not suppose that $x(t)$ is a solution to the equation.**

We start with proving the first inequality:

$$\|K(x)(t)-\xi\| = \left\| \int_\tau^t f(s, x(s))\, ds \right\| \leq \int_\tau^t \|f(s,x(s))\|\, ds \leq TM$$

Point out that it is just the eulidean norm $\|...\|$ calculated for each time point t here (no index at the norm sign) !

We observe that choosing $T < \delta/M$ we get that $\|K(x)(t)-\xi\| \leq \delta$ for $t \in [\tau, \tau+T]$. Taking supremum of the left hand side over $t \in [\tau, \tau+T]$ we arrive to

$$\|K(x)-\xi\|_{C([\tau,\tau+T])} \leq \delta$$

It means that for

$$T < \delta/M$$

the operator K maps the closed ball A in $C([\tau, \tau+T])$ defined by the inequality $\|x-\xi\|_{C([\tau,\tau+T])} \leq \delta$, into itself:

$$K : A \to A$$

Now we check conditions such that the operator K would be a contraction on the set A with once again suitably chosen time interval T.

Consider first the difference $\|K(x)(t) - K(y)(t)\|$, for arbitrary $t \in [\tau, \tau+T]$.

We apply the triangle inequality, the Lipschitz property of the function f , and estimate the integral by the length of the interval times maximum of the function under it.

$$
\begin{aligned}
\|K(x)(t)-K(y)(t)\| &= \left\| \int_\tau^t f(s,x(s))-f(s,y(s))\,ds \right\| \overset{\text{triangle inequality}}{\leq} \\
&\leq \int_\tau^t \|f(s,x(s))-f(s,y(s))\|\,ds \\
\overset{\text{Lipschitz property}}{\leq} L\int_\tau^t \|x(s)-y(s)\|\,ds &\leq \\
\overset{\sup}{\leq} LT \sup_{s\in[\tau,\tau+T]} \|x(s)-y(s)\| &= LT\,\|x-y\|_{C([\tau,\tau+T])}
\end{aligned}
$$

Calculating supremum over $t \in [\tau, \tau+T]$ of the left hand side we arrive to the inequality

$$\|K(x)-K(y)\|_{C([\tau,\tau+T])} \leq LT\,\|x-y\|_{C([\tau,\tau+T])}$$

It implies that choosing the length of the time interval

$$T < 1/L$$

we get the contraction property:

$$\|K(x)-K(y)\|_{C([\tau,\tau+T])} \leq \theta\,\|x-y\|_{C([\tau,\tau+T])}, \quad 0<\theta<1$$

Now choosing the time interval $T < \min(1/L,\, \delta/M)$ we conclude that the operator K maps the closed ball A in $C([\tau, \tau+T])$ defined by

$$A = \left\{ x \in C([\tau,\tau+T]),\ \|x-\xi\|_{C([\tau,\tau+T])} \leq \delta \right\}$$

into itself: $K : A \to A$ and that K is a contraction on A: $\|K(x)-K(y)\|_{C([\tau,\tau+T])} \leq \theta\,\|x-y\|_{C([\tau,\tau+T])}$, $\theta < 1$, for any $x, y \in A$.

By the Banach contraction principle K has for $T < \min(1/L,\, \delta/M)$ a unique fixed point $\overline{x}$ in A that is the solution to the integral equation (3) corresponding to the I.V.P. and also to the original initial value problem.$\blacksquare$

Example. Banach's contraction principle applied to a non-linear integral operator.

(exam 2019 june)

Consider the following (nonlinear!) operator

$$K(x)(t) = \int_0^2 B(t,s)\,[x(s)]^2\,ds + g(t),$$

Fixed point problem to solve:

$$x = K(x)$$

acting on the Banach space $C([0,2])$ of continuous functions with norm $\|x\|_{C([0,2])} = \|x\|_C = \sup\limits_{t\in[0,2]} |x(t)|$. Here $B(t,s)$ and $g(t)$ are continuous functions and $|B(t,s)| < 0.5$ for all $t,s \in [0,2]$.

Estimate the norm $\|K(x) - K(y)\|_{C([0,2])}$ for the operator $K(x)(t)$.

Find requirements on the function $g(t)$ such that Banach's contraction principle implies that $K(x)(t)$ has a fixed point.

Solution.

Banach's contraction principle. Let B be a nonempty closed subset of a Banach space X and let the non-linear operator $K : B \to B$ be a contraction.

$$\|K(x) - K(y)\|_X \le \theta\,\|x - y\|_X\,, \quad \theta < 1$$

Then K has a fixed point $\overline{x} = K(\overline{x})$ such that

$$\|K^n(x_0) - \overline{x}\|_X \le \frac{\theta^n}{1-\theta}$$

for any $x_0 \in B$. Here $K^n(x_0) = (K(K(...K(x_0)...))$ is the n -fold superposition of the operator K with itself.

We like to have the estimate $\|K(x) - K(y)\|_{C([0,2])} \le \theta\,\|x - y\|_{C([0,2])}$ for x, y in some closed subset B of $C([0,2])$.

$$
\begin{aligned}
|K(x)(t) - K(y)(t)| &\leq \left| \int_0^2 |B(t,s)| \left| [x(s)]^2 - [y(s)]^2 \right| ds \right| \\
&= \left| \int_0^2 |B(t,s)| \cdot |x(s) - y(s)| \cdot |x(s) + y(s)| \, ds \right| \overset{\text{taking} \quad \sup_{t,s\in[0,2]}}{\leq} \\
&\leq \int_0^2 ds \left(\sup_{t,s\in[0,2]} |B(t,s)| \right) \left(\sup_{s\in[0,2]} |x(s) - y(s)| \right) \times \\
&\quad \times \left(\sup_{s\in[0,2]} |x(s)| + \sup_{s\in[0,2]} |y(s)| \right) \\
&= 2 \cdot 0.5 \, \|x - y\|_{C([0,2])} \left(\|x\|_{C([0,2])} + \|y\|_{C([0,2])} \right) = \\
&= \|x - y\|_{C([0,2])} \left(\|x\|_{C([0,2])} + \|y\|_{C([0,2])} \right)
\end{aligned}
$$

We take supremum over $t \in [0,2]$ of the left hand side and get

$$
\|K(x) - K(y)\|_{C([0,2])} \leq \|x - y\|_{C([0,2])} \left(\|x\|_{C([0,2])} + \|y\|_{C([0,2])} \right)
$$

We can choose a ball $B \subset C([0,2])$ such that for any $x,\ y \in B$ it follows $\|x\|_C + \|y\|_C \leq \theta < 1$, for example B can be taken as a set of continuous functions with $\|x\|_{C([0,2])} \leq \theta/2$. On this set K will be a contraction because

$$
\|K(x) - K(y)\|_C \leq \theta \, \|x - y\|_C \,, \quad \theta < 1.
$$

To apply Banach's principle we need also that K maps B into itself, namely that $\|K(x)\|_{C([0,2])} \leq \theta/2$ for $\|x\|_{C([0,2])} \leq \theta/2$.

It gives a requirement on function $g(t)$.

$$
\begin{aligned}
K(x)(t) &= \int_0^2 B(t,s) \, [x(s)]^2 \, ds + g(t), \\
\|K(x)\|_{C([0,2])} &\leq 2 \times 0.5 \times \|x\|^2_{C([0,2])} + \|g\|_{C([0,2])} \leq (\theta/2)^2 + \|g\|_{C([0,2])} \leq \theta/2
\end{aligned}
$$

Conclusion is that $\|g\|_{C([0,2])} = \sup_{t\in[0,2]} |g(t)| \leq \theta/2 - (\theta/2)^2 = \theta/2 \, (1 - \theta/2)$ implies that $K : B \to B$, where

$$
B = \left\{ x(t) \in C([0,2]) : \|x(t)\|_{C([0,2])} \leq \theta/2 \right\}
$$

Therefore K has a unique fixed point in the ball B in $C([0,2])$. ■

Example. (exam. 2018 january)

1. Consider the following initial value problem: $y' = \sin(y)t^2$; $y(1) = 2$.

   a) Reduce the initial value problem to an integral equation and give a general description of iterations approximating the solution as in the proof to the existence and uniqueness theorem by Picard and Lindelöf.

   b) Find a time interval such that these approximations (Picard iterations) converge to the solution of the initial value problem. **(2p)**

   Solution.

   We introduce an integral equation equivalent to the ODE $y' = f(t, y)$ by the integration of the right and left hand sides in the equation:

$$y(t) = y(1) + \int_1^t f(s, y(s))ds.$$

   Taking $y_0(t) = y(1)$ we define Picard iterations by the recurrense relation

$$\begin{aligned} y_{n+1}(t) &= y(1) + \int_1^t f(s, y_n(s))ds. \\ y_{n+1} &= \mathbb{K}(y_n) \\ \mathbb{K}(y) &= y(1) + \int_1^t f(s, y(s))ds \end{aligned}$$

   For the particular equation it looks as

$$y_{n+1}(t) = y(1) + \int_1^t \sin(y_n(s))s^2 ds = \mathbb{K}(y_n)(t).$$

   Fixed point problem:

$$y = \mathbb{K}(y)$$

   The Banach contraction principle gives existence and uniqueness of solutions by showing that the operator $\mathbb{K}$ is a contraction:

$$\|\mathbb{K}(y) - \mathbb{K}(w)\|_X \leq \theta \|y - w\|_X; \quad \theta < 1$$

on some closed set B, $y, w \in B$, of a Banach space X, such that $\mathbb{K}$ maps B into itself.

A hidden question in the practical application, is that in applications we must find this Banach space X and this closed set B where these conditions are satisfied.

One proves the existence and uniqueness theorem by showing that at some time interval the integral operator $\mathbb{K}(y,t) = y(1) + \int_1^t \sin(y(s))s^2 \mathrm{d}s$ in the right hand side is a contraction in $C([1,T])$ for some unknown time interval $[1,T]$:

$$\begin{aligned} \|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1,T]} &\overset{def}{=} \sup_{t\in[1,T]} |\mathbb{K}(w,t) - \mathbb{K}(u,t)| \\ < \ \alpha \sup_{t\in[1,T]} |w(t) - u(t)| &= \alpha \|w - u\|_{C([1,T]} \end{aligned}$$

$\alpha < 1$, in some ball $\|w - y(1)\|_{C([1,T]} = \sup_{t\in[1,T]} |w(t) - y(1)| \leq R$ in the space $C([1,T])$ of continuous functions on $[1,T]$, and maps this ball into itself:

$$\sup_{t\in[1,T]} |\mathbb{K}(w,t) - y(1)| \leq R$$

and applying the Banach contraction theorem to $\mathbb{K}(y,t)$.

We estimate first $\|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1,T]} = \sup_{t\in[1,T]} |\mathbb{K}(w,t) - \mathbb{K}(u,t)|$

We will find T such that the contraction property is valid:

$$\begin{aligned} \|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1,T]} &= \sup_{t\in[1,T]} \left| \int_1^t \sin(w(s))s^2 ds - \int_1^t \sin(u(s))s^2 ds \right| \leq \\ &\leq \alpha \sup_{t\in[1,T]} |w(t) - u(t)|, \quad \alpha < 1 \end{aligned}$$

We carry out elementary estimates using the triangle inequality and intermediate value theorem for sin.

$$\begin{aligned} \left| \int_1^t \sin(w(s))s^2 ds - \int_1^t \sin(u(s))s^2 ds \right| &\leq \int_1^t |(\sin(w(s)) - \sin(u(s))| \, s^2 ds = \\ \int_1^t |(w(s) - u(s)) \cos(\theta(s))| \, s^2 ds &\leq (T-1)\, T^2 \cdot 1 \cdot \sup_{t\in[1,T]} |w(s) - u(s)| \end{aligned}$$

Taking supremum of the left hand side in the inequality

$$\|\mathbb{K}(w) - \mathbb{K}(u)\|_{C([1,T]} \leq (T-1)\, T^2 \, \|w - u\|_{C([1,T]}$$

The argument $\theta(s)$ above is a number between $w(s)$ and $u(s)$ that exists according the intermediate value theorem:

$$g(x) - g(y) = g'(\theta)(x - y)$$

where θ is in between x and y.

It was also used above that $|\cos(\theta)| \leq 1$. Therefore to have the contraction property we need to have $(T-1)\,T^2 < 1$.

We carry out these reasonings for continuous functions u and w such that

$$\sup_{t\in[1,T]} |w(t) - y(1)| \leq R$$

$$\|w - y(1)\|_{C([1,T]} = \sup_{t\in[1,T]} |w(t) - y(1)| \leq R.$$

Point out that $\sup_{t\in[1,T]} |w(t)| \leq y(1) + R$.

For a function w with $\|w(s) - y(1)\|_{C([1,T]} = \sup_{t\in[1,T]} |w(t) - y(1)| \leq R$ we like to have that $|\mathbb{K}(w,t) - y(1)| \leq R$ meaning that $\mathbb{K}$ maps this ball in $C([1,T])$ into itself. For this particualr case it is not necessary because the equation is defined on the whole $\not R$ and the contraction property is valid in the whole $C([1,T])$. But this checking might be necessary if the contraction property is valid only locally, not in thew whole $C([1,T])$.

The following estimate leads to another bound for T:

$$\sup_{t\in[1,T]} |\mathbb{K}(w)(t) - y(1)| \leq \sup_{t\in[1,T]} \left| \int_1^t \sin(w(s)) s^2 ds \right| \leq (T-1)\,T^2 \leq R.$$

Therefore the time interval must satisfy estimates $(T-1)\,T^2 < 1$ and $(T-1)\,T^2 < R$ to have convergence of Picard iterations in the ball $\sup_{t\in[1,T]} |w(t) - y(0)| \leq R$. Taking $R = 1$ we get an optimal estimate $(T-1)\,T^2 < 1$ that is satisfied for example for $T = 1.4$:

$$\alpha = 0.4(1.4)(1.4) = 0.784$$

■