Main ideas and tools in the course on ODE

1. Integral form of I.V.P. to ODEs
2. Grönwall's inequality for showing uniqueness and continuity with respect to data.
3. Transition mapping. Orbits of solutions, phase portrait.
4. Generalised eigenspaces of matrices. Basis of generalized eigenvectors.
5. Jordan form of matrices. Functions of matrices, in particular exponent and logarithm.
6. Transition matrix. Chapmen-Kolmogorov relations.
7. Monodromy matrix. Floquet theory for periodic linear systems.
8. Stability and instability of equilibrium points.
9. Linearization and Grobman - Hartman theorem. (iff $\mathrm{Re}(\lambda) \neq 0$)
10. Lyapunov functions (for stability, instability, and for finding positively invariant sets).
11. ω - limit sets. LaSalle's invariance principle for hunting ω - limit sets "living" in $V_f^{-1}(0)$.
12. Idea of solving integral equations by iterations (Banach's contraction priniple).

Examples of typical problems

Example on an application of Jordan matrix

For one particular solution of the system $\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t)$ with a real matrix A, the first component has the form $x_1 = t^2 + t\sin(t)$.

1. Which smallest size can the real matrix A have? **(4p)**

   Solution.

   The term $t\sin(t)$ in the solution is a sign that the Jordan form of the matrix A has a Jordan block corresponding to the eigenvalue $\lambda_1 = i$ that has multiplicity at least 2, for example $\begin{bmatrix} i & 1 \\ 0 & i \end{bmatrix}$ or multiplicity 3 : $\begin{bmatrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{bmatrix}$ etc. On the other hand te matrix A is real and therefore it's characteristic plolynomial has real coefficients and therefore all complex eigenvalues must appear as conjugate pairs: the matrix A must have the eigenvalue $\lambda_2 = -i$ havingthe same multiplicity as λ_1, at least 2 and with corresponding Jordan block $\begin{bmatrix} -i & 1 \\ 0 & -i \end{bmatrix}$.

The presence of the term t^2 in one component of a solution shows that the matrix A must have the eigenvalue $\lambda = 0$ with multiplicity at least 3 with correspoding Jordan block $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

All these observations imply that the real matrix A must have dimensions at least 7×7, because the sum of dimensions of sizes of Jordan blocks is at least $2 + 2 + 3 = 7$.■

Example of transition mapping.

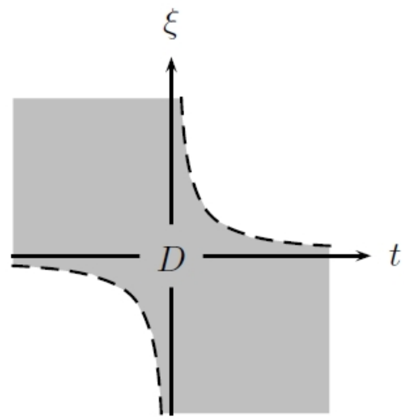
Example 4.33 of a transition map.
$G = \mathbb{R}$; $f : G \to \mathbb{R}$; $f(x) = x^2$; for $\xi = 0$; $x(t) \equiv 0$.
Initial data $x(0) = \xi$

$$\begin{aligned}
\frac{dx}{dt} &= x^2; \quad \int \frac{dx}{x^2} = \int dt; \\
-\frac{1}{x} &= t + C \\
-\frac{1}{x} &= t - \frac{1}{\xi}; \quad -\frac{1}{x} = \frac{t\xi - 1}{\xi} \\
x(t) &= \frac{\xi}{(1 - t\xi)}
\end{aligned}$$

The maximal interval for $\xi = 0$; $x(t) \equiv 0$. is $I_\xi = \mathbb{R}$
The maximal interval for $\xi > 0$, $I_\xi = (-\infty, 1/\xi)$.
The maximal interval for $\xi < 0$, $I_\xi = (1/\xi, \infty)$

$$\varphi(t, \xi) = \frac{\xi}{(1 - t\xi)}; \quad D(\varphi) = \{(t, \xi) \in \mathbb{R} \times \mathbb{R}; \quad t\xi < 1\}$$

The domain D of φ is an open set. The function $\varphi(t, \xi)$ is continuous and even locally Lipschitz.



Example of a transition mapping and maximal solutions (a bit more complicated).

1) Solve the initial value problem

$$\dot{x} = t\,x^3, \qquad x\,(1) = \xi$$

and find maximal intervals for solutions. Give a sketch of the domain for the transfer mapping $\varphi(t, 1, \xi) {=} x(t)$ in the (t, x) plane.

2) Can one conclude which maximal interval have solutions to the similar equation

$$\dot{x} = t^3 x$$

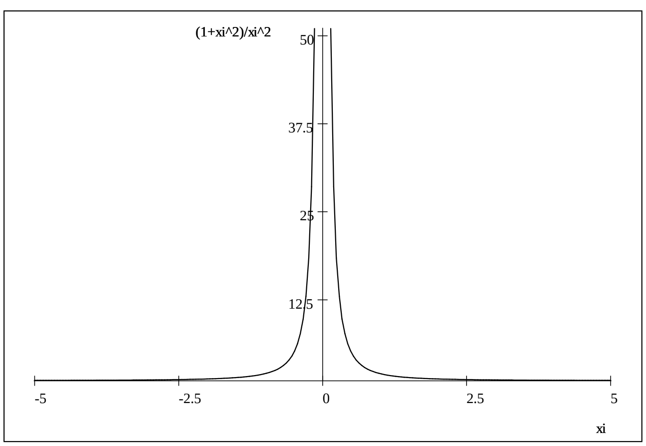
without solving it?

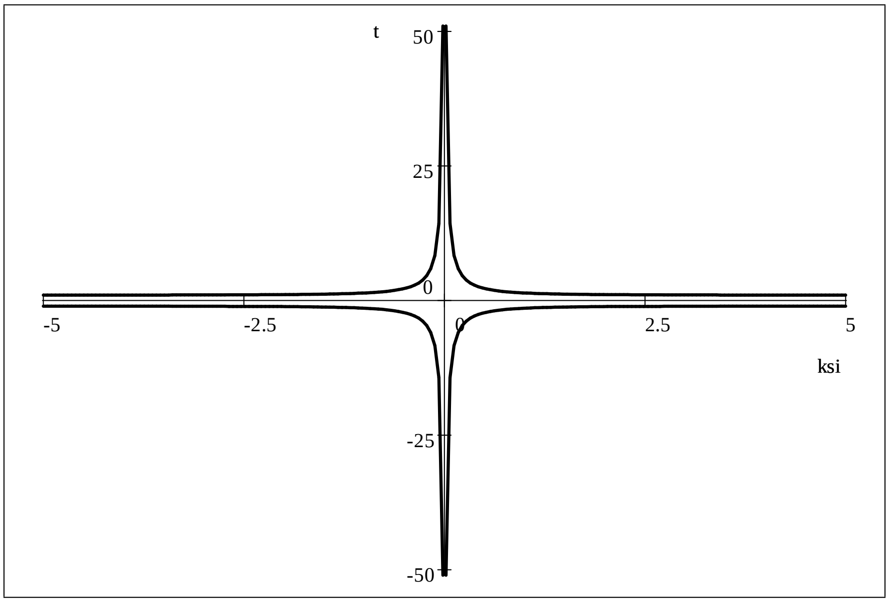
Solution.

1) It is the equation with separable variables.

$$\begin{aligned}
\frac{dx}{dt} &= tx^3; \qquad x(1)=\xi \\
\int \frac{dx}{x^3} &= \int t\,dt \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - C \\
C &= \frac{t^2}{2} + \frac{1}{2x^2}; \quad C = \frac{1}{2} + \frac{1}{2\xi^2} = \frac{1+\xi^2}{2\xi^2} \\
\frac{-1}{2x^2} &= \frac{t^2}{2} - \frac{1+\xi^2}{2\xi^2} \\
\frac{-1}{2x^2} &= \frac{\xi^2 t^2}{2\xi^2} - \frac{1+\xi^2}{2\xi^2} = \frac{\xi^2 t^2 - \left(1+\xi^2\right)}{2\xi^2} \\
x^2 &= \frac{\xi^2}{\left(1+\xi^2\right) - \xi^2 t^2} = \frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2} \\
x &= \sqrt{\frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2}}, \left(1+\xi^2\right)/\left(\xi^2\right) - t^2 > 0, \xi > 0 \\
x &= -\sqrt{\frac{1}{\left(1+\xi^2\right)/\left(\xi^2\right) - t^2}}, \left(1+\xi^2\right)/\left(\xi^2\right) - t^2 > 0, \xi < 0 \\
x &\equiv 0, \quad \xi = 0, \quad -\text{equilibrium}, \quad t \in \mathbb{R} \\
\left(1+\xi^2\right)/\left(\xi^2\right) &> t^2; \quad t \in \left(-\sqrt{\left(1+\xi^2\right)/\left(\xi^2\right)}, \sqrt{\left(1+\xi^2\right)/\left(\xi^2\right)}\right) OPEN!!!
\end{aligned}$$

1. The maximal intervals for these solutions are open in accordance with the general theory. One solution $x \equiv 0$ is defined on the whole $\mathbb{R}$. We draw boundaries of the domain for $\varphi(t, 1, \xi)$.





Example of an equation with "eternal" solutions.

The equation $\dot{x} = t^3 x$ is defined on $\mathbb{R} \times \mathbb{R}$ and the right hand side satisfies on any compact time interval $[-R, R]$, $R > 0$ the estimate $\left|t^3 x\right| \leq R^3(1 + |x|)$ where the right hand side rises linearly with respect to $|x|$. It implies that the maximal existence interval for all solutions to this equation is $\mathbb{R}$.

Estimating Lyapunov functions V and their derivatives $V_f = \nabla V \cdot f$ along trajectories.

Investigation of the sign of functions V and $V_f = \nabla V \cdot f$.

Choosing a Lyapunov's function for stability analysis: it must be positive definite: $V(0) = 0$, $V(x) > 0$, $x \neq 0$.

This property lets to use some of the level sets also as boundaries for 1) positively invariant sets and 2) regions (or domains) of attraction for asymptotycally stable equilibrium points.

(For instability analysis it is enough to find a test function V such tat it is positive arbitrarily close to the equilibrium point in the origin, for example on a line through the origin or in a cone with the vertex in the origin).

The second step in finding Lyapunovs functions is consideration of the sign of the function $V_f(x) = \nabla V \cdot f(x)$. This finction gives the rate of change for $V(x)$ trajectories $x(t)$ of the differential equation $x' = f(x)$ without solving the equation, because $\frac{d}{dt} V(x(t)) = \nabla V \cdot f(x(t))$.

The choice of test functions

1. The simplest choice of a test function V for using in Lyapunovs theorems is $V(x,y) = x^2+y^2$ having level sets being circles around the origin. It is often our first choice. Sometimes test functions like $V(x,y) = ax^2+bxy+cy^2$ with indefinite terms xy can be convenient if they are positive definite.

2. Test functions as a sum of kinetic and potential energy.One dimensional Newton equation. First integrals

For systems in the form

$$\begin{aligned} x' &= y, \\ y' &= -ay - g(x) \end{aligned}$$

defined for all $(x,y) \in \mathbb{R}^2$ equivalent to the Newton equation

$$x'' = -ax' - g(x),$$

with potential force $-g(x)$ it is natural and optimal to choose a test function as a sum of the kinetic energy $\frac{1}{2}y^2$ and $G(x) = \int_0^x g(s)ds$:

$$V(x,y) = \frac{1}{2}y^2 + \int_0^x g(s)ds$$

If the force is an odd function such that $xg(x) > 0$, $x \neq 0$, and $g(0) = 0$ this test function $V(x,y)$ will be positive definite in some region around the origin.

The derivative V_f of V along trajectories for the friction force equal to $-ay$, $a > 0$

$$\begin{aligned} (\nabla V \cdot f)(x,y) &= \left(\frac{\partial}{\partial x}V\right) f_1 + \left(\frac{\partial}{\partial y}V\right) f_2 \\ &= g(x)y + x(-g(x)) - ay^2 = -ay^2 \leq 0 \end{aligned}$$

The Lyapunov stability theorem would imply that the origin is a stable equilibrium point. Depending on how the potential $G(x) = \int_0^x g(s)ds$ behaves and on the position of other equilibrium points (zeroes of the function $g(x)$), one can find a region of attraction bounded by a level set of V that includes only one equilibrium point.

One can use the same idea in the case when the friction force in the equation above has the form: $-a\phi(y)$ with $\phi(y)y > 0$,

2. Test functions as a higher order polynomial arbitrary even powers and with arbitrary coefficients.

A flexible choice of a test function $V(x,y)$ can be

$$V(x,y) = ax^m + by^n$$

with arbitrarty exponents m, n and arbitrary coefficients $a, b > 0$ that are chosen after the calculation of $V_f(x,y)$ so that $V_f(x,y) \leq 0$ or $V_f(x,y) \leq 0$ for $(x,y) \neq (0,0)$.

Example: $V(x,y) = x^2 + xy + 2y^2$. Level sets of such a test function will be ellipses with the axis rotated with respect to the coordinate system. The Cauchy inequality

$$|xy| \leq \frac{1}{2}\left(x^2 + y^2\right)$$

helps to show that this test function is positie definite.

A more general Young inequality

$$|ab| \leq \frac{a^p}{p} + \frac{b^q}{q}; \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 1$$

can be useful for investigating polynomials of higher degree in f:

This property $V(x) > 0,\ x \neq 0,\ V(0) = 0$ is a condition in the theorem by Lyapunov on stability. It implies i particular that level sets of V close to the origin are closed curves.

Analysis of V_f

We like to have $V_f = \nabla V \cdot f(x)$ negative definite $V_f(x) < 0$ or at least $\nabla V \cdot f(x) \leq 0$ for $x \neq 0$.

Here f is the right hand side ("velocity") in the differential equation of interest: $x' = f(x)$. It makes $\frac{d}{dt}(x(t)) = \nabla V \cdot f(x(t))-$ showing how the test function V changes along trajectories $x(t)$.

Analysis of $V_f^{-1}(0)$

After calculating $V_f(x)$ we check the set $V_f^{-1}(0)$ where $V(x) = 0$. Why it is interesting?

The La Salle's invariance principle states that all ω - limit sets of trajectories $x(t)$ inside the domain where $\nabla V \cdot f(x) \leq 0$ is valid belong to the set $V_f^{-1}(0)$ and they belong even to a smaller part of it that is the maximal invariant subset in $V_f^{-1}(0)$.

How to apply La Salle's invariance principle ?

i) The set $V_f^{-1}(0)$ is easy to identify, as a set of zeroes to V_f (in plane in most of our examples). It is usually one or both coordinate axes.

ii) The maximal invariant set inside $V_f^{-1}(0)$ (in the plane it will be a set of curves) is easy to check invariant sets just by looking on velocities (values of $f(x,y)$) on the set $V_f^{-1}(0)$ and checking if they go along curves forming $V_f^{-1}(0)$ or they go across.

It implies in particular that if in addition to the inequality $\nabla V \cdot f(x)$ ≤ 0 the set $V_f^{-1}(0)$ includes only an invariant set consisting of the origin, then, the origin is asymptotically stable equilibrium.

Example.

Consider the following system of ODE: $\begin{cases} x' = & -x - 2y + xy^2 \\ y' = & 3x - 3y + y^3 \end{cases}$.

Show asymptotic stability of the equilibrium point in the origin and find the region of attraction for that.

Hint: applying Lyapunovs theorem, use the elementary inequality

$$|xy| \le \frac{1}{2}\left(x^2+y^2\right)$$

to estimate possible indefinite terms with xy in the expression for $V_f(x,y)$.

Solution. Choose a test function $V(x,y) = \frac{1}{2}\left(x^2+y^2\right)$
$V_f = \nabla V \cdot f = x(-x-2y+xy^2)+y\left(3x-3y+y^3\right) = xy-x^2-3y^2+y^4+x^2y^2$
$= -x^2\left(1-y^2\right)-y^2\left(3-y^2\right)+\underbrace{xy}_{indefinite\_term!} \le -x^2\left(1-y^2\right)-y^2\left(3-y^2\right)+$
$0.5x^2+0.5y^2$

We apply the inequality $2xy \le \left(x^2+y^2\right)$ to the last term and collecting terms with x^2 and y^2 arrive to the estimate

$V_f \le -x^2\left(0.5-y^2\right) - y^2\left(2.5-y^2\right)$

It implies that $V_f < 0$ for $(x,y) \neq (0,0)$ and $|y| < 1/\sqrt{2}$.Therefore the Lyapunof function V is "strong" and therefore the origin is asymptotically stable.

The region of attraction is bounded by the largest levle set of V - a circle having the center in the origin that fits to the domain $|y| < 1/\sqrt{2}$, namely the circle: $\left(x^2+y^2\right) < 1/2$.

The second idea for choosing Lyapunov functions is choice of V of polynomilas with arbitrary even powers and arbitrary coefficients.

Another more clever choice of a test function as

$$V(x,y) = ax^m + by^n$$

in particular $V(x,y) = 3x^2+2y^2$ works in this particular case:

$V_f = 6x(-x-2y+xy^2)+4y(3x-3y+y^3) = 4y^4-12y^2-6x^2+6x^2y^2 = -4y^2\left(3-y^2\right) - 6x^2\left(1-y^2\right) < 0$

for $|y| < 1$, therefore the ellipse $3x^2+2y^2 < 2$ that fits into the stripe $|y| < 1$ in the plane is a region of attraction for the asymptotically stable equilibrium in the origin.

One can also observe the asymptotic stability of the origin here by linearization with variational matrix

$A = \begin{bmatrix} -1 & -2 \\ 3 & -3 \end{bmatrix}$, with characteristic polynomial: $\lambda^2+4\lambda+9=0$, and calculating eigenvalues: $-i\sqrt{5}-2, i\sqrt{5}-2$ with $\operatorname{Re}\lambda < 0$. But linearization gives no information about the domain of attraction.

Application of Poincare - Bendixson theorem

The generalized Poincare-Bendixson's theorem gives a complete description of possible types of ω - limit sets in the plane $\mathbb{R}^2$.

Theorem (generalized Poincare-Bendixson)

Let M be an open subset of $\mathbb{R}^2$ and $f: M \to \mathbb{R}^2$ and $f \in C^1$. Fix $\xi \in M$ and suppose that the closure of $\Omega(\xi) \neq \emptyset$, is compact, connected and contains only finitely many equilibrium points.

In practice it is enough and is much easier to find a compact positively invariant set $K \subset M$ such that $\xi \in K$.

Then one of the following cases holds:

(i) $\Omega(\xi)$ is an equilibrium point

(ii) $\Omega(\xi)$ is a periodic orbit

(iii) $\Omega(\xi)$ consists of finitely many fixed points $\{x_j\}$and non-closed orbits γ such that ω and α - limit points of γ belong to $\{x_j\}$.

Poincare - Bendixson theorem and testing the absence of equilibrium points in a positive invariant set.

We try to find an ring shaped compact set that is positively invariant and need to check three conditions:

i) The outer boundary of the ring (using a level set of a test function, or a polygon shaped domain testing velosities on each segment of it's boundary)

ii) The inner boundary of the ring (using a level set of a test function, or linearization for identifying a repeller inside a large postively invariant set by applying the Grobman - Hartman theorem)

iii) Show that the found compact positively invariant ring shaped set includes no equilibrium points. (this condition is often missed by students)

1. Consider the following system of ODEs. $\begin{cases} x' = & y \\ y' = & -x - y\left[\ln\left(x^2+4y^2\right)\right] \end{cases}$.

   Show that this system has a non-trivial periodic solution. **(4p)**

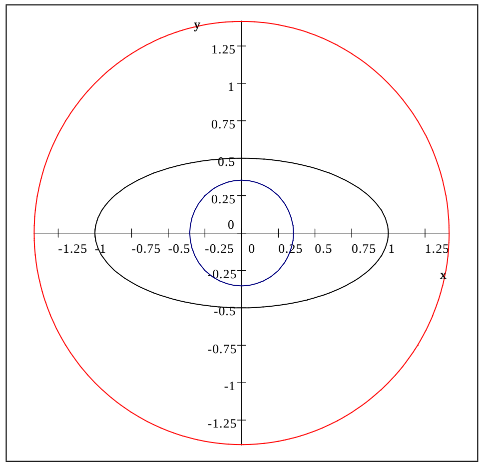
   Point out that the origin is outside the domain of the equation.

   Solution.

   Consider the test function $E(x,y) = \frac{1}{2}\left(x^2+y^2\right)$

   $\frac{d}{dt}E(x(t),y(t)) = E_f(x,y) = \nabla E \cdot f(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} y \\ -x - y\left[\ln\left(x^2+4y^2\right)\right] \end{bmatrix} =$
   $-y^2\left[\ln\left(x^2+4y^2\right)\right]\begin{cases} \geq 0 & 0 < x^2+4y^2 < 1 \\ \leq 0 & x^2+4y^2 > 1 \end{cases}$

   The boundary curve separating domains with different signs of $x^2+4y^2 = 1$



is the ellipse with halv axes 1 and 1/2 i the x - direction with center in the origin.

Therefore any circle with the center in the origin inside this ellipse is never entered by a trajectory.

Similarly any circle with the center in the origin outside this ellipse is never left by a trajectory.

Such two circles build an annulus that is a compact positively invariant set for this system of ODEs.

For example an annulus $1/4 \leq x^2 + y^2 \leq 1$ satisfies this conditions. This annulus contains no equilibrium points, because the origin is the only equilibrium point. Therefore by the Poincare - Bendixson theorem this annulus must contain at least one periodic orbit.■

Example. Show that the following system of ODEs has a periodic solution.

$$\begin{cases} x' = x - 2y - x\left(2x^2 + y^2\right) \\ y' = 4x + y - y\left(2x^2 + y^2\right) \end{cases} \qquad \textbf{(4p)}$$

Solution. Consider the following test function: $V(x, y) = 2x^2 + y^2$. Denoting the right hand side in the equation by vectorfunction $F(x, y)$ we conclude that

$V_f = \nabla V \cdot f = 4x^2 - 8xy - 4x^2\left(2x^2 + y^2\right) + 8xy + 2y^2 - 2y^2(2x^2 + y) = 2\left(1 - (2x^2 + y^2)\right)(2x^2 + y^2)$.

It implies that the elliptic shaped ring: $R = \left\{(x, y) : 0.5 \leq (2x^2 + y) \leq 2\right\}$ is a positive invariant compact set for the ODE, because velocity vectors are directed inside of this ring both on it's outer and inner boundaries ($\nabla V \cdot F < 0$ for $(2x^2 + y) = 2$ and $\nabla V \cdot F > 0$ for $(2x^2 + y) = 0.5$.

The origin is the only equilibrium point of the system**. It is not so easy to see from the system of equations itself.** But one can see it easier by cheching first zeroes of $V_f(x, y)$ that is a scalar function and evidently must be zero in all equilibrium points..

We observe that $V(x,y) = 2x^2 + y^2$ is positive definite and $\nabla V \cdot f(x,y) = 0$ only if $(x,y) = (0,0)$ or if $(2x^2+y^2) = 1$.But it is easy to see from the expression for the right hand side for the ODE that in the last case (x,y) cannot be equilibrium point because the right hand side becomes linear with nondegenerate matrix and is zero only in the origin $(x,y) = (0,0)$. The equation for equilibrium points on the level set $(2x^2 + y^2) = 1$ is the following:

$$\begin{cases} 0 = x - 2y - x = -2y \\ 0 = 4x + y - y = 4x \end{cases}$$

By the Poincare-Bendixson theorem the positively invariant set R not including any equilibrium point must include at least one orbit of a periodic solution.■

Problem on ω - limit sets(January 2020)

Consider the following system of ODEs. $\begin{cases} x' = & y \\ y' = & x - x^3 - ay\left(y^2 - x^2 + \frac{1}{2}x^4\right), \quad a > 0 \end{cases}$.

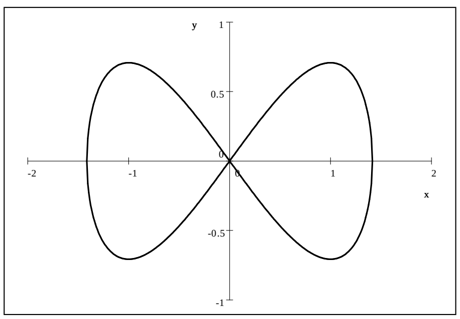
1. Find all systems equilibrium points. Show using the test function $H = \frac{1}{2}\left(y^2 - x^2 + \frac{1}{2}x^4\right)$ and La Salle's invariance principle, that the level set $H(x,y) = 0$ includes ω - limit sets of this system for all points in the plane except a finite number. Sketch these ω - limit sets. **(4p)**

   Solution.

   The system has three equilibrium points, all on the $x-$axis: $(-1,0)$, $(0,0)$, $(1,0)$. The level set $H(x,y) = \frac{1}{2}\left(y^2 - x^2 + \frac{1}{2}x^4\right) = 0$ has the shape of ∞ with the center in the origin. One can see it by expressing y in terms of x:

   $$y = \pm |x| \sqrt{1 - \frac{1}{2}x^2}$$

   The ∞ figure is symmetric with respect to x - axis and cuts it in points $\pm\sqrt{2}$. The formula above implies that $H(x,y) > 0$ outside of the ∞ figure, and $H(x,y) < 0$ inside of the ∞ figure.



We calculate how the H function changes along trajectories.

$$H_f(x,y) = \frac{d}{dt}H(x(t),y(t)) = \begin{bmatrix} -x+x^3 \\ y \end{bmatrix} \cdot \begin{bmatrix} y \\ x - x^3 - ay\left(y^2 - x^2 + \frac{1}{2}x^4\right) \end{bmatrix} =$$

$$\underbrace{-xy + x^3y + xy - x^3y}_{=0} - ay^2 \underbrace{\left(y^2 - x^2 + \frac{1}{2}x^4\right)}_{H(x,y)}$$

We point out that $\frac{d}{dt}H(x(t),y(t)) = 0$ on the level set $H(x,y) = 0$ (the ∞ figure) and on the x - axis. It means that trajectories are tangential to the level set $H(x,y) = 0$. Therefore ∞ - figure is an invariant set for the system and consists of three orbits: the equilibrium in the origin (that is a saddle point, easily seen by linerization) and two closed branches of the ∞ figure correspoding to $x > 0$ and $x < 0$ in the expression $y = \pm|x|\sqrt{1 - \frac{1}{2}x^2}$.

$H_f(x,y) = \frac{d}{dt}H(x(t),y(t)) < 0$ outside of the ∞ figure and not on the x - axis where $\frac{d}{dt}H(x(t),y(t)) = 0$.

$H_f(x,y) = \frac{d}{dt}H(x(t),y(t)) > 0$ inside of the ∞ figure and not on the x - axis where $\frac{d}{dt}H(x(t),y(t)) = 0$.

By La Salle's invariance principle all trajectories are attracted to the largest invariat set inside the set $H_f^{-1}(0)$,were $H_f(x,y) = 0$. This set consists of the union of the ∞ figure and the x - axis. There are no invariant sets on the x - axis except three equilibrium points $(-1,0)$, $(0,0)$, $(1,0)$.

It implies that for all points in the plain except equilibrium points, and points on the ∞ figure, $H(x(t),y(t))$ tends to zero along trajectories. The ω - limit sets for these points consist of one of the branches of the ∞ figure (for points inside it) or of the whole ∞ figure - for points outside it. The origin is the ω - limit set for all points on the ∞ figure. Equilibrium points are ω - limit sets of themselfs

Problem on stability of equilibrium points and on domains of attraction.

Consider the following system of ODEs. $\begin{cases} x' = & 1 - xy \\ y' = & x - y^3 \end{cases}$

Find all equilibrium points and investigate their stability. Find domains of attraction for possible asymptotically stable equilibrium points. **(4p)**

Solution.

Equilibrium points are $(1,1)$ and $(-1,-1)$ can be found by substitution. $x = y^3$, $1 = xy = y^4$.

Jacoby matrix of the right hand side is $J(x,y) = \begin{bmatrix} -y & -x \\ 1 & -3y^2 \end{bmatrix}$; $J(1,1) = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$; $J(-1,-1) = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$. $\det(J(1,1)) = 4$, $\mathrm{tr}(J(1,1)) = -4$. Therefore the equilibrium point $(1,1)$ is asymptotically stable.

$\det(J(-1,-1)) = -4$. Therefore the linearized around $(-1,-1)$ system has a saddle point and the equilibrium point $(-1,-1)$ is unstable.

We shift the origin of the coordinate system into the point $(1,1)$ by introducing new variables $u = x-1, v = y-1$ and $x = u+1$, $y = v+1$.

$$\begin{cases} u' = & -u - v - uv \\ v' = & u - 3v - 3v^2 - v^3 \end{cases}$$

Consider a test function $E(u,v) = \frac{1}{2}\left(u^2+v^2\right)$

$$\begin{aligned} \frac{d}{dt}E(u(t),v(t)) &= \begin{bmatrix} u \\ v \end{bmatrix} \cdot \begin{bmatrix} -u-v-uv \\ u-3v-3v^2-v^3 \end{bmatrix} = \\ &= -u^2 - uv - u^2v + uv - 3v^2 - 3v^3 - v^4 = \\ &= -u^2(1-v) - 3v^2\underbrace{\left(1+v+v^2\right)}_{>0} < 0 \\ if \quad v &< 1, \quad (u,v) \neq (0,0) \end{aligned}$$

The largest circle in (u,v) plane satisfying the condition $v \leq 1$ has radius 1. Therefore the circle of radius 1 around the equilibrium point $(1,1)$ is the domain of attraction for the asymptotically stable equilibrium $(1,1)$ of the original system of ODEs.■