6th Lecture: 27/1

First (algebraic) proof of Theorem 5.10: Let $G(x) = \sum_{n=0}^{\infty} C_n x^n$. Observe that $xG(x) = \sum_{n=0}^{\infty} C_n x^{n+1} = \sum_{n=1}^{\infty} C_{n-1} x^n$. Now,

$$\begin{aligned}
[xG(x)][G(x)] &= \left[\sum_{n=1}^{\infty} C_{n-1} x^n\right] \left[\sum_{n=0}^{\infty} C_n x^n\right] = \\
&= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} C_{m-1} C_{n-m}\right) x^n \overset{(5.13)}{=} \sum_{n=1}^{\infty} C_n x^n \\
&= \sum_{n=0}^{\infty} C_n x^n - C_0 x^0 = G(x) - 1.
\end{aligned}$$

Thus

$$x[G(x)]^2 - G(x) + 1 = 0.$$

This can be considered a quadratic equation for $G(x)$ and hence, using the usual formula for the roots of a quadratic equation[1],

$$G(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.$$

Note that if we took the plus sign on the RHS, then it would not converge as $x \to 0$. So the minus sign must be correct and we conclude that

$$G(x) = \frac{1}{2x} \left(1 - (1 - 4x)^{1/2}\right).$$

We now apply the Generalized Binomial Theorem, eq. (5.7) in the notes, to get

$$G(x) = \frac{1}{2x} \left(1 - \sum_{k=0}^{\infty} \binom{1/2}{k} (-4x)^k\right) = -\sum_{k=1}^{\infty} \binom{1/2}{k} \frac{(-4)^k}{2} x^{k-1}.$$

By definition, C_n is the coefficient of x^n in this expression, hence

$$\begin{aligned}
C_n = -\binom{1/2}{n+1} \frac{(-4)^{n+1}}{2} &= -\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \dots \left(-\frac{2n-1}{2}\right)}{(n+1)!} \, \frac{(-1)^{n+1} 4^{n+1}}{2} \\
&= (-1) \times \frac{(-1)^n \times 1 \times 3 \times \cdots \times (2n-1)}{2^{n+1} (n+1)!} \times \frac{(-1)^{n+1} 4^{n+1}}{2} \\
&= \frac{1 \times 3 \times \cdots \times (2n-1)}{(n+1)!} \times 2^n.
\end{aligned}$$

[1]This formula is derived by completing squares, i.e.: by pure algebraic manipulation of the equation. These manipulations are valid in any *field*, hence in particular in the field of formal power series $\mathbb{C}[[x]]$, in which the function $G(x)$ lies.

Now the trick is to multiply above and below by $2^n\, n!$ and note that $2^n\, n! = 2 \times 4 \times 6 \times \cdots \times 2n$. Thus,

$$\begin{aligned} C_n &= [1 \times 3 \times \cdots \times (2n-1)][2 \times 4 \times \cdots \times 2n] \times \frac{2^n}{(2^n\, n!) \times (n+1)!} \\ &= \frac{(2n)!}{n!\,(n+1)!} = \frac{1}{n+1}\,\frac{(2n)!}{n!\,n!} = \frac{1}{n+1}\binom{2n}{n}, \quad \text{v.s.v.} \end{aligned}$$

SECOND (COMBINATORIAL) PROOF OF THEOREM 5.10: First note that

$$\binom{2n}{n-1} = \frac{(2n)!}{(n-1)!\,(n+1)!} = \frac{(2n)!}{n!\,n!} \times \frac{n}{n+1} = \binom{2n}{n} \times \frac{n}{n+1},$$

so in order to prove Theorem 5.10, it suffices to prove that

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}. \tag{6.1}$$

Note that $\binom{2n}{n}$ is the total number of rightward diagonal paths from $(0,\, 0)$ to $(2n,\, 0)$, as already noted in Remark 5.12. Hence we need to show that exactly $\binom{2n}{n-1}$ of these paths go under the x-axis at least once. To accompish this, we will describe one-to-one correspondences between sets of paths as follows (all paths are assumed to be directed rightwards):

$$\begin{aligned} &\{\text{Diagonal paths from } (0,\, 0) \text{ to } (2n,\, 0) \text{ which go under the } x\text{-axis at least once}\} \\ &\Updownarrow \\ &\{\text{Diagonal paths from } (0,\, 1) \text{ to } (2n,\, 1) \text{ which meet the } x\text{-axis at least once}\} \\ &\Updownarrow \\ &\{\text{All diagonal paths from } (0,\, -1) \text{ to } (2n,\, 1)\}. \end{aligned}$$

Note that this will suffice: the third set clearly contains $\binom{2n}{n-1}$ paths, since any path in this set contains $n+1$ up-steps and $n-1$ down-steps. The 1-1 correspondence from the first to the second set is also simple to describe, namely just move an entire path one unit upwards. The clever idea is how to get a 1-1 correspondence between the second and third sets.

The idea is called *Andre's Reflection Principle*. Consider a diagonal path from $(0,\, 1)$ to $(2n,\, 1)$ which meets the x-axis at least once. Let $(k,\, 0)$ be the point at which it meets the x-axis for the first time. Now take the part of the path from $(0,\, 1)$ to $(k,\, 0)$ and *reflect* it in the x-axis. This procedure yields a diagonal path from $(0,\, -1)$ to $(2n,\, 1)$, namely the reflection of the original path up to $(k,\, 0)$ followed by the original path thereafter. It is then just a matter of realising that *every* diagonal path from $(0,\, -1)$ to $(2n,\, 1)$ can be uniquely obtained in this manner, i.e.: by Andre reflection of a unique diagonal path $(0,\, 1) \to (2n,\, 1)$ which meets the x-axis at least once.

The ideas are summarized in Figure 6.1. $\square$

Multivariable recurrences. We will give some examples of recurrences involving doubly-infinite sequences, that is sequences indexed by two variables $(u(i,\, j))_{i,j=0}^{\infty}$. Indeed, one can find examples with an arbitrary number of variables. In general, rigorous mathematical analysis becomes more difficult as the number of variables is increased, with the jump from one to two variables already constituting a quantum leap. This is roughly analogous to the jump from one- to multi-variable calculus, or from ODEs to PDEs. In this course, we just give a few examples which arise naturally in a combinatorial context and in connection to things we have already discussed in earlier lectures. As well as the examples below, see the discussion on Ramsey numbers in Lecture Z.

Example 6.1. Let $(c_{n,\,k})_{n,\,k=0}^{\infty}$ be defined by

$$c(0,\,0) = 1, \quad c(0,\,k) = 0 \;\forall\; k > 0, \quad c(n,\,k) = c(n-1,\,k) + c(n-1,\,k-1) \;\forall n \geq 1.$$

If one stares for long enough, one recognises the above as the defining relations for the binomial coefficients $C(n,\,k)$, the recurrence being Pascal's identity (Proposition 2.4). Thus $c(n,\,k) = C(n,\,k) = \frac{n(n-1)\dots(n-k+1)}{k!}$. I leave it as an optional exercise to derive this formula "systematically" by using the bivariate generating function

$$G(x,\,y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c(n,\,k)\, x^n\, y^k.$$

In Lecture 1, we introduced the study of "Balls and Bins" and derived formulae (Propositions 1.10 and 1.11) for the number of ways to place balls into *distinguishable* bins. We now turn to the case where the bins are indistinguishable. It turns out that this case is much harder, and the best we can do is derive fairly simple-looking (but not so simple so as to yield nice explicit formulae) recurrences for the number of ways to distribute the balls in such a way that *no bin is left empty*. There are two cases, depending on whether the balls are distinguishable or not. The first case is Example 6.2 below, the second will be Example 6.7.

Example 6.2. The *Stirling number of the second kind* $S(n,\,k)$ is defined to be the number of ways to distribute n distinguishable balls among k indistinguishable bins in such a way that no bin is left empty. Clearly, the last requirement implies that $S(n,\,k) = 0$ whenever $k > n$. For other cases, we have the following result, which is Theorem 5.9 in Volume 1 of the course book:

Theorem 6.3.

$$S(n,\,1) = S(n,\,n) = 1,$$
$$S(n,\,k) = S(n-1,\,k-1) + k \cdot S(n-1,\,k), \quad \textit{whenever}\; 2 \leq k \leq n-1. \tag{6.2}$$

PROOF: $S(n,\,1) = 1$ since if we have only one bin then it must receive every ball. Similarly, $S(n,\,n) = 1$ since if we have as many bins as balls, and no bin is to be left empty, then each bin must receive exactly one ball. It doesn't matter which bin gets which ball, since the bins are indistinguishable.

Now consider a general pair $(n,\,k)$. Since the balls are distinguishable, we can isolate a particular ball, call it b, and consider two cases:

Case 1: Ball b is placed in a bin on its own. It doesn't matter which bin, since the latter are identical. It then remains to distribute the remaining $n-1$ balls among $k-1$ bins while not leaving any bin empty. By definition, this can be done in $S(n-1,\,k-1)$ ways.

Case 2: Ball b is not on its own. In this case it matters where we place ball b - since the balls are distinguishable, it matters which balls it is placed together with. So we begin by distributing the remaining $n-1$ balls into the k bins in such a way that no bin is left empty. This can be done in $S(n-1,\, k)$ ways. There are then k choices for the bin to receive ball b and, since every bin already has at least one ball, it matters where we put ball b. By MP, there are a total of $k \cdot S(n-1,\, k)$ ways to distribute the n balls in Case 2.

Finally, an application of AP yields (6.2).

Remark 6.4. Just like the numbers $\binom{n}{k}$, the numbers $S(n,\, k)$ can be visualised in a triangle. From (6.2) we see that, just like for Pascal's triangle, there will be ones along the edges. This time, each other number is a weighted sum of the two diagonally above, namely that to the right is weighted by its position along that row. Google an image if you like !

Remark 6.5. Recall that an *equivalence relation* on a set X is a subset $\mathcal{R}$ of $X \times X$ satisfying the following three properties:
Reflexivity: $(x,\, x) \in \mathcal{R}$ for all $x \in X$. In words, every element of x is related to itself.
Symmetry: $(x,\, y) \in \mathcal{R} \Leftrightarrow (y,\, x) \in \mathcal{R}$. In words, x is related to y if and only if y is related to x.
Transitivity: If $(x,\, y) \in \mathcal{R}$ and $(y,\, z) \in \mathcal{R}$ then $(x,\, z) \in \mathcal{R}$. In words, if x is related to y and y to z, then x is related to z.

An equivalence relation on a set X gives rise to a partition of the set into so-called *equivalence classes*, such that two elements are related if and only if they lie in the same class.

We can now give a further interpretation of the Stirling numbers $S(n,\, k)$, namely: $S(n,\, k)$ is the number of different equivalence relations on an n-element set which give rise to exactly k equivalence classes.

Remark 6.6. Recall from Proposition 1.10 that k^n is the number of ways to distribute n distinguishable balls amongst k *distinguishable* bins and that this is the same as the number of functions from an n-set to a k-set (see Example 1.2). Now it is also immediate from the definition that $k!\, S(n,\, k)$ is the number of ways to distribute n distinguishable balls amongst k distinguishable bins *in such a way that no bin is left empty*. Hence, this is also the number of *surjective* functions from an n-set to a k-set. Thus Theorem 6.3 gives an alternative means of computing the number of such functions, compared to the Inclusion-Exclusion principle (Example 2.12).

Example 6.7. The number of ways to distribute n identical balls into k identical bins such that no bin is left empty is denoted $p(n,\, k)$, and is usually referred to as the number of *partitions of n into exactly k parts*. There are a number of different recurrences for partition numbers, perhaps the simplest is

$$p(n,\,k) = p(n-1,\,k-1) + p(n-k,\,k), \tag{6.3}$$

which is an exercise on Homework 1. We can also note some special cases:

(i) $p(n,\,1) = 1$ since the only partition into one part is n itself.

(ii) $p(n,\,n) = 1$ since the only paritition of n into n parts is $1+1+\cdots+1$ (n times).

(iii) $p(n,\,n-1) = 1$ since the only paritition of n into $n-1$ parts is to have one part equal to 2 and all other parts equal to 1, i.e.: $2+1+\cdots+1$.

(iv) $p(n,\,2) = \lfloor n/2 \rfloor$ since every partition of n into two parts is of the form $k+(n-k)$, where $\lceil n/2 \rceil \leq k \leq n-1$.

To summarise (i)-(iv):

$$p(n,\,1) = p(n,\,n) = p(n,\,n-1) = 1, \quad p(n,\,2) = \lfloor \frac{n}{2} \rfloor. \tag{6.4}$$

The function

$$p(n) = \sum_{k=1}^{n} p(n,\,k)$$

has been extensively studied in Number Theory. It is the total number of partitions of n, or just the "partition function". A famous problem, which was essentially solved in 1918 by Hardy and Ramanujan (and independently by Uspensky in 1920), was to determine a good asymptotic estimate for $p(n)$. Their result is that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}\,n}, \tag{6.5}$$

where $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. Note that the growth rate of $p(n)$ is superpolynomial but subexponential, an indication of a highly non-trivial behaviour since, for example, it follows from previous lectures that sequences satisfying linear recurrences always exhibit either polynomial or exponential growth. To do justice to the rich theory on partitions is beyond the scope of this course. See wiki. For the record, we just list all partitions of 8:

$$\begin{aligned}
&8, \quad 7+1, \quad 6+2, \quad 6+1+1, \quad 5+3, \quad 5+2+1, \\
&5+1+1+1, \quad 4+4, \quad 4+3+1, \quad 4+2+2, \quad 4+2+1+1, \\
&4+1+1+1+1, \quad 3+3+2, \quad 3+3+1+1, \quad 3+2+2+1, \quad 3+2+1+1+1, \\
&3+1+1+1+1+1, \quad 2+2+2+2, \quad 2+2+2+1+1, \quad 2+2+1+1+1+1, \\
&2+1+1+1+1+1+1, \quad 1+1+1+1+1+1+1+1.
\end{aligned}$$

Hence $p(8) = 22$ and, breaking it down,

$$\begin{gathered}
p(8,\,1) = 1, \quad p(8,\,2) = 4, \quad p(8,\,3) = 5, \quad p(8,\,4) = 5, \\
p(8,\,5) = 3, \quad p(8,\,6) = 2, \quad p(8,\,7) = 1, \quad p(8,\,8) = 1.
\end{gathered}$$