HOMEWORK 1 – DUE MAY 4 AT 15:15

This homework sheet consists of 10 questions, each worth 5 points. Hand in solutions either in person (e.g. at the lecture) or via email to `jakobbj at chalmers` *in* `pdf` *format; you can use the free apps* `CamScanner` *or* `Genius Scan` *to make pdf scans using your phone.*

(1) Let $X_1, X_2, \dots$ be independent Bernoulli random variables with $\mathbb{P}(X_n = 1) = 1 - \mathbb{P}(X_n = 0) = p_n$. Show that
  (a) $X_n \to 0$ in probability, if and only if $p_n \to 0$;
  (b) $X_n \to 0$ almost surely, if and only if $\sum_{n\geq 1} p_n < \infty$.

(2) For random variables X, Y on $(\Omega, \mathcal{F})$, define a distance $d(X, Y) = \mathbb{E}[1 \wedge |X - Y|]$. (Here $\wedge$ means minimum.) Prove that X_n converge to X in probability if and only if $d(X_n, X) \to 0$.

(3) Let X, Y be random variable with $\mathbb{E}|X| < \infty$ and $\mathbb{E}|Y| < \infty$ which satisfy
$$\mathbb{E}[X \mid Y] = Y \qquad \text{and} \qquad \mathbb{E}[Y \mid X] = X.$$
Show that $X = Y$ almost surely.
Hint: consider quantities like $\mathbb{E}[(X - Y)\mathbb{1}_{\{X>c, Y\leq c\}}] + \mathbb{E}[(X - Y)\mathbb{1}_{\{X\leq c, Y\leq c\}}]$.

(4) Let X be uniformly chosen in $[0, 1]$ and let $X = 0.X_1X_2X_3\dots$ be its binary expansion. Let Y_n be the number of consecutive 0's starting from position n. We saw in the lectures that the X_i are independent with $\mathbb{P}(X_i = 1) = \frac{1}{2}$ and that
$$\mathbb{P}\big(\limsup_{n\to\infty} \tfrac{Y_n}{\log_2 n} \leq 1\big) = 1.$$
In what follows, r_n is a non-decreasing sequence of positive real numbers.
  (a) Show that the event $A = \{Y_n \geq r_n \text{ i.o.}\}$ has probability 0 or 1.
  (b) Show that if $\sum_{n\geq 1} 2^{-r_n} < \infty$ then $\mathbb{P}(A) = 0$.
  (c) Show that if $\sum_{n\geq 1} 2^{-r_n}/r_n = \infty$ then $\mathbb{P}(A) = 1$.
  (d) Deduce that
$$\mathbb{P}\big(\limsup_{n\to\infty} \tfrac{Y_n}{\log_2 n} = 1\big) = 1.$$

(5) Consider a stochastic process $X(t)$ defined by
$$X(t) = U\cos(t) + V\sin(t), \qquad t \in \mathbb{R},$$
where U and V are independent random variables taking the values -2 or 1 with probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Show that $X(t)$ is weakly stationary but not strongly stationary.

(6) Let $k \geq 2$ and let $Z_1, Z_2, \ldots, Z_k$ be i.i.d. integrable random variables with positive variance. Define a process $\mathbf{X} = (X_n)_{n \geq 0}$ by

$$X_{rk+j} = Z_{j+1}, \qquad r \in \{0, 1, 2, \ldots\}, \quad j \in \{0, 1, \ldots, k-1\}.$$

In what follows you may assume that $\mathbf{X}$ is defined on a probability space of your choosing.

(a) Show that $\mathbf{X}$ is a strictly stationary process which is *not* ergodic (i.e. the measure $\mathbb{Q}(A) = \mathbb{P}(\mathbf{X} \in A)$ is not ergodic).

(b) Find an explicit formula for $\mathbb{E}[X_1 \mid \mathcal{I}]$ and show (without using the ergodic theorem) that it satisfies the definition of conditional expectation given $\mathcal{I}$.

(7) (a) Determine the spectral distribution of a stationary stochastic process of the form

$$X(t) = \sum_{k=0}^{\infty} C_k e^{-i\lambda_k t}, \qquad t \in \mathbb{R},$$

where the C_k are independent complex random variables with $\mathbb{E}[C_k] = 0$ and $\mathbb{E}[|C_k|^2] = \sigma_k^2$, where $\sum_{k \geq 0} \sigma_k < \infty$, and where the λ_k are fixed real numbers.

(b) Find a stationary stochastic process $Y(t)$ whose spectral distribution is Poisson with mean μ.

(8) Consider the following 'paintbox-construction' of a random sequence $\mathbf{X} \in \{0,1\}^\infty$. We are given $p \in [0,1]$ and a sequence of numbers $\alpha_1, \alpha_2, \ldots \geq 0$ with $\sum_{i \geq 1} \alpha_i = 1$, which we use to construct $\mathbf{X}$ in two steps: (i) the indices $1, 2, 3, \ldots$ are partitioned into 'boxes' by putting i into box $\#k$ with probability α_k, independently for all $i \geq 1$; (ii) then all indices i in the same box get the same label $X_i = 0$ or $= 1$, these labels being chosen independently for different boxes, with probability p for 1.

(a) Prove that the resulting sequence $\mathbf{X}$ is exchangeable.

(b) Determine the distribution of the random variable ξ in de Finetti's Theorem in the case when there are just two boxes, i.e. $\alpha_1 = \alpha = 1 - \alpha_2$.

(c) Determine an expression for ξ in general, in terms of the α_k and some auxiliary random variables which you are to specify.

(9) A restaurant owner budgets \$100 per day for maintainance, which ensures that the restaurant is never in violation of health-and-safety regulations, but wonders if this amount can be reduced. Inspectors visit the premises on average every 45 days. Suppose they discover a health-and-safety violation during their visit with probability p, which then leads to a fine which is uniformly distributed between \$10 and \$1000. (Thus p is a function of the daily budget satisfying $p = 0$ if the budget is at least \$100.) Making the necessary assumptions about independence, determine a condition on the daily budget which would lead to savings in the long run.

(10) A ride-share scheme operates as follows. Cars, each with a capacity of 4 passengers, stand in a long line. Customers arrive, with independent inter-arrival times X_i, and take a seat in the car at the head of the line. Once that car is full it departs, and the process is then repeated with the next car in line. Denote by $X(t) \in \{0, 1, 2, 3\}$ the number of passengers waiting inside the car at time t, by $T_n = X_1 + \cdots + X_n$ the time for the arrival of the n:th customer, and by $S_n = T_{4n}$ the time at which the n:th car departs.

(a) Obtain a renewal-type equation expressing $p_j(t) = \mathbb{P}(X(t) = j)$ in terms of $q_j(t) = \mathbb{P}(X(t) = j, S_1 > t)$ and the renewal-function $m(t)$ for the car-departure-process.

(b) In the case of exponentially distributed X_i with parameter λ, compute the Laplace-transform

$$\hat{p}_j(\theta) = \int_0^\infty e^{-\theta t} p_j(t)\, dt.$$