HOMEWORK 2 – DUE MAY 25 AT 15:15

This homework sheet consists of 10 questions, each worth 5 points. Hand in solutions in person, or via email to `jakobbj at chalmers` *in* `pdf` *format; you can use the free apps* ***CamScanner*** *or* ***Genius Scan*** *to make pdf scans using your phone.*

(1) Let $(Y_n)_{n\geq 0}$ be a martingale with $\mathbb{E}(Y_n^2) < \infty$ for all n. Show that
$$\sup_{n\geq 0} \mathbb{E}(Y_n^2) < \infty \qquad \Leftrightarrow \qquad \sum_{n\geq 0} \mathbb{E}[(Y_{n+1} - Y_n)^2] < \infty.$$

(2) Let $(X_n)_{n\geq 1}$ be a sequence of independet random variables satisfying
$$\mathbb{P}(X_n = -n^2) = 1 - \mathbb{P}\big(X_n = \tfrac{n^2}{n^2-1}\big) = \tfrac{1}{n^2}.$$
Show that $S_n = X_1 + \cdots + X_n$ is a martingale such that $S_n/n \to 1$ a.s. and $S_n \to \infty$ a.s. *[Hint: Borel–Cantelli]*

(3) Let $X_1, \dots, X_n$ be independent uniformly chosen elements of a finite set S, let $(a_1, \dots, a_k)$ be some fixed sequence of elements of S of length $k \geq 2$, and let N_n be the number of occurrences of the sequence $(a_1, \dots, a_k)$ as consecutive elements of $(X_1, \dots, X_n)$.
 (a) Compute $\mathbb{E}(N_n)$ (for example by considering the events A_i that the chosen sequence starts at position i).
 (b) Assuming that $|S| \geq \sqrt{n}$ and that k is fixed, show that $N_n/n^{1/2+\delta}$ converges to 0 in probability for any $\delta > 0$. *[Hint: Hoeffding]*.

(4) Consider a game with $N \geq 1$ rounds, where your winnings per unit stake on round n are i.i.d. random variables X_n satisfying
$$\mathbb{P}(X_n = 1) = 1 - \mathbb{P}(X_n = -1) = p, \qquad p \in (\tfrac{1}{2}, 1).$$
Your stake S_n on round n should be measureable with respect to $\mathcal{F}_{n-1} = \sigma(X_1, \dots, X_{n-1})$ (i.e. predictable) and be smaller than Z_{n-1}, where Z_{n-1} denotes your fortune at time $n-1$. Assume that Z_0 is some positive constant and write $h = -p\log(p) - (1-p)\log(1-p) - \log(2)$.
 (a) Show that $Y_n = \log(Z_n) + nh$ is a supermartingale.
 (b) What strategy maximizes the expected 'interest rate' $\mathbb{E}[\log(Z_N/Z_0)]$?

(5) Let $(Y_n)_{n\geq 0}$ be a non-negative martingale and T a finite stopping time. Show, *without appealing to the Optional Stopping Theorem (12.5.1)*, that
 (a) if there is some M such that $|Y_n| \leq M$ a.s. for all $n \geq 0$, then $\mathbb{E}(Y_T) = \mathbb{E}(Y_0)$,
 (b) if $\mathbb{P}(T < \infty) = 1$ then $\mathbb{E}(Y_T) \leq \mathbb{E}(Y_0)$.

(6) Consider successive tosses of a coin having probability p of landing heads. Use a martingale argument to compute the expected number of tosses until the following sequences occur:
  (a) `HHTTHHT`,
  (b) `HTHTHTH`.

(7) Let $(Y_n)_{n\geq 0}$ be a sequence of random variables in $[0,1]$ defined as follows. First $Y_0 = \alpha$ for some fixed $\alpha \in (0,1)$. Then, given Y_n, we have

$$Y_{n+1} = \begin{cases} \frac{Y_n}{2}, & \text{with probability } 1 - Y_n, \\ \frac{Y_n+1}{2}, & \text{with probability } Y_n. \end{cases}$$

  (a) Prove that $(Y_n)_{n\geq 0}$ is a martingale with respect to the filtration given by $\mathcal{F}_n = \sigma(Y_0, \ldots, Y_n)$, and that it converges in L^p for every $p \geq 1$ to some Y_∞.
  (b) Verify that $\mathbb{E}[(Y_{n+1} - Y_n)^2] = \frac{1}{4}\mathbb{E}[Y_n(1-Y_n)]$ and use this to determine $\mathbb{E}[Y_\infty(1-Y_\infty)]$ and hence the law of Y_∞.

(8) In this problem we will prove de Finetti's theorem using a martingale-approach. Let $(X_1, X_2, \ldots)$ be an exchangeable sequence with $X_i \in \{0,1\}$. Write

$$P_{n,r} = X_1^r + \cdots + X_n^r$$

and define

$$\mathcal{G}_n = \sigma(P_{n,1}, P_{n,2}, \ldots, P_{n,n}, X_{n+1}, X_{n+2}, \ldots)$$

and $\mathcal{G}_\infty = \bigcap_{n\geq 1} \mathcal{G}_n$.
  (a) Show that $\mathbb{E}(X_1 \mid \mathcal{G}_n)$ defines a backward martingale and deduce that $\frac{1}{n}\sum_{i=1}^n X_i \to Y_\infty := \mathbb{E}[X_1 \mid \mathcal{G}_\infty]$ almost surely and in L^p for any $p \geq 1$.
  (b) Show that $\left(\frac{1}{n}\sum_{i=1}^n X_i\right)^2 \to \mathbb{E}[X_1 X_2 \mid \mathcal{G}_\infty]$ almost surely and in L^p for any $p \geq 1$, and deduce that $\mathbb{E}[X_1 X_2] = \mathbb{E}[Y_\infty^2]$.
  (c) Show that

$$\mathbb{E}[X_1 \cdots X_k(1 - X_{k+1}) \cdots (1 - X_{k+\ell})] = \mathbb{E}[Y_\infty^k (1 - Y_\infty)^\ell]$$

  for any $k, \ell \geq 1$.

(9) Let $(B(t))_{t\geq 0}$ denote a standard Brownian motion and $0 \leq s < t$. What is the conditional distribution of
  (a) $B(t)$ given $B(s)$?
  (b) $B(s)$ given $B(t)$?

(10) Let $(B(t))_{t\in[0,1]}$ be a standard Brownian motion, and consider the two processes

$$X(t) = B(t) - tB(1), \qquad \text{and} \qquad Y(t) = \begin{cases} (1-t)B\left(\frac{t}{1-t}\right), & \text{for } t \in [0,1), \\ 0, & \text{for } t = 1. \end{cases}$$

  (a) Show that both are Gaussian processes.
  (b) Determine their covariance functions.
  (c) Show that both are almost surely continuous on $[0,1]$.