SOLUTIONS TO HOMEWORK 2

(1) We saw in the lectures that $\mathbb{E}\big[(Y_{n+1} - Y_n)Y_j \mid \mathcal{F}_n\big] = 0$, for all $j \leq n$. Writing $Y_n = Y_0 + \sum_{j=1}^n (Y_j - Y_{j-1})$ we get

$$Y_n^2 = Y_0^2 + \Big(\sum_{j=1}^n (Y_j - Y_{j-1})\Big)^2 + 2Y_0 \sum_{j=1}^n (Y_j - Y_{j-1}).$$

Thus

$$\mathbb{E}(Y_n^2) = \mathbb{E}(Y_0^2) + \mathbb{E}\Big[\Big(\sum_{j=1}^n (Y_j - Y_{j-1})\Big)^2\Big] = \mathbb{E}(Y_0^2) + \sum_{j=1}^n \mathbb{E}\big[(Y_j - Y_{j-1})^2\big],$$

since the cross-terms give 0. The right-hand-side is increasing in n, thus

$$\sup_{n\geq 0} \mathbb{E}(Y_n^2) = \mathbb{E}(Y_0^2) + \sum_{j=1}^\infty \mathbb{E}\big[(Y_j - Y_{j-1})^2\big].$$

(2) Since $\sum_{n\geq 1} \mathbb{P}(X_n = -n^2) < \infty$, we have $\mathbb{P}(X_n = -n^2 \text{ i.o.}) = 0$ meaning that, with probability 1, eventually all X_n take value $n^2/(n^2-1) \to 1$. So $X_n \to 1$ a.s. which implies $S_n/n \to 1$ a.s. and therefore $S_n \to \infty$. To see that S_n is a martingale use independence and that $\mathbb{E}[X_n] = 0$ for all n.

(3) (a) With A_i the event that the given sequence starts in position i, we have

$$\mathbb{E}(N_n) = \mathbb{E}\Big[\sum_{i=1}^{n-k+1} \mathbb{1}_{A_i}\Big] = \frac{n-k+1}{|S|^k}.$$

(b) If $|S| \geq \sqrt{n}$ and $k \geq 2$ then $\mathbb{E}(N_n)$ is bounded. Changing one coordinate X_i changes N_n by at most k. Hoeffding applied to the martingale $Z_k = \mathbb{E}[N_n \mid X_1, \dots, X_k]$ (for $k = 0, \dots, n$) gives

$$\mathbb{P}(|N_n - \mathbb{E}(N_n)| \geq x) \leq 2\exp\big(-\tfrac{x^2}{2nk^2}\big).$$

Thus for any $\varepsilon > 0$,

$$\mathbb{P}(|N_n - \mathbb{E}(N_n)| \geq \varepsilon n^{1/2+\delta}) \leq 2\exp\big(-\tfrac{\varepsilon^2 n^{2\delta}}{2k^2}\big) \to 0.$$

Since $\mathbb{E}(N_n)$ is bounded this proves the claim.

(4) Since $Z_{n+1} = Z_n + S_{n+1}X_{n+1}$ we have

$$\mathbb{E}(\log Z_{n+1} \mid \mathcal{F}_n) = p\log(Z_n + S_{n+1}) + (1-p)\log(Z_n - S_{n+1}) = \log Z_n + f(S_{n+1}/Z_n)$$

where $f(x) = p\log(1+x) + (1-p)\log(1-x)$. Now $f(x)$ is maximal for $x = 2p-1$ and $f(2p-1) = -h$, which shows that Y_n is a supermartingale, and in fact a martingale if we set $S_{n+1} = (2p-1)Z_n$ for each n. The latter is the optimal strategy.

(5) (a) For any n, the stopping time $T \wedge n$ is bounded, so by Theorem 12.4.11 we have $\mathbb{E}[Y_{T\wedge n}] = \mathbb{E}[Y_0]$. Next, $Y_{T\wedge n} \to Y_T$ a.s. and since Y is bounded, the bounded convergence theorem implies that $\mathbb{E}[Y_{T\wedge n}] \to \mathbb{E}[Y_T]$, which gives the result.

(b) By Fatou, since $Y_{T\wedge r} \to Y_T$ almost surely as $r \to \infty$,

$$\mathbb{E}(Y_T) = \mathbb{E}(\lim_{r\to\infty} Y_{T\wedge r}) \le \liminf_{r\to\infty} \mathbb{E}(Y_{T\wedge r}) = \mathbb{E}(Y_0).$$

(6) This is similar to Problem 12.9.16 in GS, so it helps to understand that first. Here we use similar notation. So at each time n a new gambler G_n enters and starts betting on the pattern `HHTTHHT` until failure. When a gambler's bet fails, the casino gains the gambler's initial fortune of \$1. Until then the fortune of the gambler is $p^{-\#\mathtt{H}}q^{-\#\mathtt{T}}$ where $\#\mathtt{H}$ and $\#\mathtt{T}$ indicate the number of heads/tails that were correct so far (and $q = 1-p$), meaning that the casino has paid out an amount $p^{-\#\mathtt{H}}q^{-\#\mathtt{T}} - 1$ to the gambler. Write S_n for the gain (or loss) of the casino so far, this is a martingale as checked in GS.

At the time N of the first `HHTTHHT` we have seen

$$* * * \cdots * {\color{red}*}\mathtt{HHTTHHT}$$

where the $*$-sequence contains no `HHTTHHT` and the red $*$ is in position $N-7$. This means that at time N, gambler G_{N-6} is still winning, as is gambler G_{N-2}. The others have all lost. So the casino has got $N-2$ from the gamblers who lost, but also paid $p^{-4}q^{-3} - 1 + p^{-2}q^{-1} - 1$ to the two who are still winning. Thus $S_N = N - p^{-4}q^{-3} - p^{-2}q^{-1}$. Below we check in detail that Theorem 12.5.1 applies meaning that we have $\mathbb{E}(S_N) = \mathbb{E}(S_0) = 0$. Assuming this we get, for (a), $\mathbb{E}(N) = p^{-4}q^{-3} + p^{-2}q^{-1}$.

Similarly, for (b) we have

$$* * * \cdots * {\color{red}*}\mathtt{HTHTHTH}$$

and then gamblers $N-6$, $N-4$, $N-2$ and N are still winning. Thus $S_N = N - (p^{-4}q^{-3} + p^{-3}q^{-2} + p^{-2}q^{-1} + p^{-1})$ and $\mathbb{E}(N) = p^{-4}q^{-3} + p^{-3}q^{-2} + p^{-2}q^{-1} + p^{-1}$.

Now we verify that Theorem 12.5.1 applies (one could also verify 12.5.9 instead). Writing $r = \mathbb{P}(N = 7) = p^4q^3 > 0$, there is probability r that the given sequence occurs in tosses 1 to 7, in tosses 8 to 14, in tosses 9 to 21, etc. This means that $\mathbb{P}(N > 7k) \le (1-r)^k$ for any $k \ge 0$. Then

$$\mathbb{E}(N) = \sum_{n\ge 0} \mathbb{P}(N > n) \le \sum_{k\ge 0} 7\mathbb{P}(N > 7k) < \infty.$$

In particular $\mathbb{P}(N < \infty) = 1$. Combined with our expressions for S_N this also verifies that $\mathbb{E}|S_N| < \infty$. And note that if $n < N$ then gamblers $1, 2, \dots, n-6$ have all lost while $n-5, n-4, \dots, n$ may (in principle, depending on the pattern) still be winning. Those that are still winning have a fortune of at most $p^{-\#\mathtt{H}}q^{-\#\mathtt{T}} \le (pq)^{-6}$ so a very rough bound is that

$$\mathbb{E}(|S_n|\mathbb{1}\{n < N\}) \le [n - 6 + 6(pq)^{-6}]\mathbb{P}(N > n)$$

which by our previous bounds on $\mathbb{P}(N > 7k)$ gives that $\mathbb{E}(|S_n|\mathbb{1}\{n < N\}) \to 0$ as required.

This method can be used to compute the expected time until a monkey, hitting random letters on a keyboard, accidentally writes Shakespeare's play Hamlet. A more reasonable version of this is to compute the expected time until the monkey types ABRACADABRA.

(7) (a) If we write

$$Y_{n+1} = \tfrac{Y_n}{2}\mathbb{1}\{Y_{n+1} = \tfrac{Y_n}{2}\} + \tfrac{1+Y_n}{2}\mathbb{1}\{Y_{n+1} = \tfrac{1+Y_n}{2}\}$$

then we get

$$\begin{aligned}\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] &= \tfrac{Y_n}{2}\mathbb{E}[\mathbb{1}\{Y_{n+1} = \tfrac{Y_n}{2}\} \mid \mathcal{F}_n] + \tfrac{1+Y_n}{2}\mathbb{E}[\mathbb{1}\{Y_{n+1} = \tfrac{1+Y_n}{2}\} \mid \mathcal{F}_n] \\ &= \tfrac{Y_n}{2}\mathbb{P}(Y_{n+1} = \tfrac{Y_n}{2} \mid \mathcal{F}_n) + \tfrac{1+Y_n}{2}\mathbb{P}(Y_{n+1} = \tfrac{1+Y_n}{2} \mid \mathcal{F}_n) \\ &= \tfrac{Y_n}{2}(1 - Y_n) + \tfrac{1+Y_n}{2}Y_n = Y_n.\end{aligned}$$

It is bounded between 0 and 1 and hence converges a.s. and in L^p for all $p \geq 1$.

(b) Expanding and using the martingale property we get

$$\mathbb{E}\big[(Y_{n+1} - Y_n)^2\big] = \mathbb{E}\big[\mathbb{E}[(Y_{n+1} - Y_n)^2 \mid \mathcal{F}_n]\big] = \mathbb{E}\big[\mathbb{E}[Y_{n+1}^2 \mid \mathcal{F}_n] - Y_n^2\big].$$

Similar calculations as for (a) give

$$\mathbb{E}[Y_{n+1}^2 \mid \mathcal{F}_n] = \tfrac{1}{4}(3Y_n^2 + Y_n).$$

Hence $\mathbb{E}[(Y_{n+1} - Y_n)^2] = \frac{1}{4}\mathbb{E}[Y_n(1 - Y_n)]$.

The left-hand-side converges to 0 and the right-hand-side to $\frac{1}{4}\mathbb{E}[Y_\infty(1-Y_\infty)]$. It follows that

$$Y_\infty(1 - Y_\infty) = 0 \quad \text{a.s.}$$

This means that Y_∞ has values in $\{0, 1\}$. But $\mathbb{E}[Y_\infty] = \mathbb{E}[Y_0] = \alpha$, hence Y_∞ takes value 1 with probability α and 0 otherwise.

(8) (a) The sequence $Y_n = \mathbb{E}[X_1 \mid \mathcal{G}_n]$ is a backward martingale since the $\mathcal{G}_n$ are decreasing. To see this, we can either note that $P_{n,r} = P_{n,1}$ for all $r \geq 1$ since the X_i take values 0 or 1 only, or (which applies for more general X_i) that $P_{n-1,1}, \dots, P_{n-1,n-1}$ generate all symmetric polynomials in $X_1, \dots, X_{n-1}$, in particular we can write $P_{n,n} = P_{n-1,n} + X_n^n$ using them together with X_n.
Being a backward martingale, $Y_n \to Y_\infty$ almost surely. Since it is bounded the convergence also holds in L^p. Next, exchangeability implies that $Y_n = \mathbb{E}[X_i \mid \mathcal{G}_n]$ for any $i = 1, 2, \dots, n$ and therefore

$$Y_n = \mathbb{E}\big[\frac{1}{n}\sum_{i=1}^n X_i \mid \mathcal{G}_n\big] = \mathbb{E}\big[\tfrac{1}{n}P_{n,1} \mid \mathcal{G}_n\big] = \frac{1}{n}\sum_{i=1}^n X_i,$$

the last step following from $\mathcal{G}_n$-measurability. The claim follows.

(b) By exchangeability, for $n \geq 2$,

$$\begin{aligned}\mathbb{E}[X_1X_2 \mid \mathcal{G}_n] &= \mathbb{E}\big[\frac{1}{n(n-1)}\sum_{1\leq i\neq j\leq n} X_iX_j \mid \mathcal{G}_n\big] \\ &= \frac{1}{n(n-1)}\mathbb{E}\big[\big(\sum_{i=1}^n X_i\big)^2 - \sum_{i=1}^n X_i^2 \mid \mathcal{G}_n\big] \\ &= \frac{1}{n(n-1)}\Big[\big(\sum_{i=1}^n X_i\big)^2 - \sum_{i=1}^n X_i^2\Big].\end{aligned}$$

Here, on the left-hand-side, $\mathbb{E}[X_1X_2 \mid \mathcal{G}_n]$ is a backward martingale converging to $\mathbb{E}[X_1X_2 \mid \mathcal{G}_\infty]$. The right-hand-side equals $\big(\frac{1}{n}\sum_{i=1}^n X_i\big)^2$ up to a correction which $\to 0$ almost surely as $n \to \infty$. We conclude that

$$\big(\frac{1}{n}\sum_{i=1}^n X_i\big)^2 \to \mathbb{E}[X_1X_2 \mid \mathcal{G}_\infty].$$

But we also know that
$$\Big(\frac{1}{n}\sum_{i=1}^{n} X_i\Big)^2 \to Y_\infty^2.$$
Hence $\mathbb{E}[X_1X_2 \mid \mathcal{G}_\infty] = Y_\infty^2$. Taking expectation gives the claim.

(c) This is similar to the previous part. First note that it suffices to prove the statement for $\ell = 0$ (by expanding and using exchangeability). Then note that, for $n \geq k$,
$$\begin{aligned}\mathbb{E}[X_1 \cdots X_k \mid \mathcal{G}_n] &= \mathbb{E}\Big[\tfrac{1}{n(n-1)\cdots(n-k+1)} \sum_{1\leq i_1 \neq \cdots \neq i_k \leq n} X_{i_1}\cdots X_{i_k} \mid \mathcal{G}_n\Big] \\ &= \Big(\tfrac{1}{n}\sum_{i=1}^{n} X_i\Big)^k + o(1).\end{aligned}$$
This gives $\mathbb{E}[X_1 \cdots X_k \mid \mathcal{G}_\infty] = Y_\infty^k$ and hence the claim.

(9) (a) Since $B(t) = B(s) + [B(t) - B(s)]$ and the increments are independent and normally distributed, the conditional distribution of $B(t)$ given $B(s)$ is $\mathrm{N}(B(s),\, t-s)$.

(b) Use the time-inversion property to see that the pair $(B(s), B(t))$ has the same distribution as the pair $(\tilde{B}(s), \tilde{B}(t)) = (sB(\frac{1}{s}), tB(\frac{1}{t}))$. We have
$$\tilde{B}(s) = sB(\tfrac{1}{s}) = s\big(B(\tfrac{1}{t}) + [B(\tfrac{1}{s}) - B(\tfrac{1}{t})]\big) = \tfrac{s}{t}\tilde{B}(t) + s[B(\tfrac{1}{s}) - B(\tfrac{1}{t})],$$
where the term $[B(\frac{1}{s}) - B(\frac{1}{t})]$ is independent of $\tilde{B}(t)$ and normally distributed with mean 0 and variance $\frac{1}{s} - \frac{1}{t}$. So the conditional distribution of $B(s)$ given $B(t)$ is $\mathrm{N}(\frac{s}{t}B(t),\, \frac{s}{t}(t-s))$.

(10) (a) Since B is a Gaussian process, any linear combination of terms $B(t_i)$ is normally distributed. Now any linear combination of terms $X(t_i)$ or terms $Y(t_i)$ reduces to a linear combination of terms $B(t_i)$, and is hence normally distributed. Thus X and Y are both Gaussian processes.

(b) Assume $s < t$. Then a simple calculation gives
$$\mathrm{Cov}(X(s), X(t)) = \mathrm{Cov}(Y(s), Y(t)) = s(1-t).$$

(c) It is clear for X, and for Y continuity is clear on $[0, 1)$. To prove that $Y(t)$ is almost surely continuous at $t = 1$, write $T = \frac{1}{1-t}$, then
$$\mathbb{P}\big(\lim_{t\uparrow 1} Y(t) = 0\big) = \mathbb{P}\big(\lim_{T\uparrow\infty} \tfrac{1}{T}B(T-1) = 0\big) = 1$$
by Corollary 1.11 in Mörters–Peres.

Both X and Y are the 'Brownian bridge', which may also be interpreted as standard BM conditioned to hit 0 at $t = 1$.