MOCK EXAMINATION 2022 Examiner:
Stochastic Processes MVE330/MSF200 J. Björnberg

The actual exam will consist of 6–8 questions for a total maximum of 50 points. Any bonus points are added on top, and grading is as follows. CTH: 20, 30, 40 points for 3, 4, 5 respectively. GU: 20, 35 points for G, VG respectively. PhD-students: 20 points for pass.

No tools allowed (pen is OK). **Good luck!**

1. Let $X_1, X_2, X_3, \ldots$ be i.i.d. with $\mathbb{P}(X_i = 1) = 1 - \mathbb{P}(X_i = -1) = p$ and let $S_n = X_1 + \ldots + X_n$ (with $S_0 = 0$).

   (a) Show that $\{S_n = 0 \text{ i.o.}\}$ is not a tail-event for the X_i. (2p)

   (b) Show that if $p \neq \frac{1}{2}$ then $\mathbb{P}(S_n = 0 \text{ i.o.}) = 0$. (2p)

   (c) Show that for any $x > 0$, the event

   $$\big\{ \liminf_{n\to\infty} \tfrac{S_n}{\sqrt{n}} \leq -x \big\} \cap \big\{ \limsup_{n\to\infty} \tfrac{S_n}{\sqrt{n}} \geq x \big\}$$

   is a tail-event, and deduce that if $p = \frac{1}{2}$ then $\mathbb{P}(S_n = 0 \text{ i.o.}) = 1$. (2p)

2. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of integrable random variables.

   (a) Define a *stopping time* with respect to this sequence. (1p)

   (b) State and prove *Wald's identity.* (2p)

   (c) Assume now that the X_i satisfy $\mathbb{P}(X_i > 0) = 1$ and let $N(t) = \max\{n \geq 1 : X_1 + \cdots + X_n \leq t\}$ be a renewal process. Show that $T = N(t) + 1$ is a stopping time, for any $t > 0$. (2p)

   (d) Give an example of a random time T, measurable with respect to the sequence $X_1, X_2, \ldots$, such that Wald's identity does not hold for the time T. (2p)

3. This problem goes through a proof of the ergodic theorem which is a bit different from that in the lectures and the book. *Remark: this question is certainly at the hard end of the spectrum for exam questions...*

   Let $X_1, X_2, \ldots$ be a strongly stationary and ergodic sequence such that $\mathbb{E}|X_1| < \infty$ and $\mu = \mathbb{E}[X_1] > 0$. Let $S_n = X_1 + \cdots + X_n$ and $J_n = \min_{1\leq k\leq n} S_k$.

   (a) Show that $\mathbb{E}[J_n] = \mu + \mathbb{E}[\min\{0, J_{n-1}\}]$ for any $n \geq 2$. (2p)

   (b) Let $x^+ = \max\{0, x\}$ denote the positive part of x. Deduce from (a) that

   $$\sum_{k=1}^{n} \mathbb{E}[J_k^+] = n\mu + \mathbb{E}[J_1 - J_{n+1}] \qquad \text{for } n \geq 2,$$

   and hence that $\mathbb{E}[J_n^+] \geq \mu$ for any $n \geq 2$. (3p)

   (c) Deduce that $\mathbb{P}(\inf_{n\geq 1} S_n > -\infty) = 1$. (2p)

   (d) Use this to establish that $\frac{1}{n} S_n \to \mu$ almost surely as $n \to \infty$. (2p)

Continued →

4. Let $N(t)$ be a renewal-process whose inter-arrival times are uniformly distributed in $[0,1]$.

   (a) Write down the renewal equation for $m(t) = \mathbb{E}[N(t)]$ and use it to show that $m(t) = e^t - 1$ for $t \in [0,1]$. (2p)

   (b) Work out a renewal-type equation for the second moment $m_2(t) = \mathbb{E}[N(t)^2]$ and use it to find a formula for the variance of $N(t)$ for $t \in [0,1]$. (3p)

5. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

   (a) Define the following terms: *martingale*, *submartingale* and *predictable process* (also called *previsible*). (3p)

   (b) Assume that $(X_n)_{n\geq 0}$ is an integrable submartingale. Define a process $(Z_n)_{n\geq 0}$ by $Z_0 = 0$ and $Z_n = Z_{n-1} + \mathbb{E}[X_n \mid \mathcal{F}_{n-1}] - X_{n-1}$. Show that $(Z_n)_{n\geq 0}$ is predictable, and conclude that any integrable submartingale can be written as a sum of an increasing predictable process and a martingale. (3p)

   (c) Assume that $(Y_n)_{n\geq 0}$ is a martingale such that $Y_0 = 0$ and $\mathbb{E}(Y_n^2) < \infty$ for all $n \geq 0$, and let $X_n = Y_n^2$. Show that $(X_n)_{n\geq 0}$ is a submartingale, and that the corresponding predictable process $(Z_n)_{n\geq 0}$ may be written as

   $$Z_n = \sum_{m=1}^{n} \mathbb{E}[(Y_m - Y_{m-1})^2 \mid \mathcal{F}_{m-1}]. \qquad (3p)$$

   (d) Under the same assumptions as the previous item, show that Z_n has an a.s. limit Z_∞ (possibly taking the value ∞) and that $\mathbb{E}[\sup_{n\geq 0} Y_n^2] \leq 4\mathbb{E}[Z_\infty]$.
   Hint: use Doob's L^r-inequality. (3p)