MOCK EXAMINATION 2022 Examiner:
Stochastic Processes MVE330/MSF200 J. Björnberg

The actual exam will consist of 6–8 questions for a total maximum of 50 points. Any bonus points are added on top, and grading is as follows. CTH: 20, 30, 40 points for 3, 4, 5 respectively. GU: 20, 35 points for G, VG respectively. PhD-students: 20 points for pass.

No tools allowed (pen is OK). **Good luck!**

1. This is Exercise 7.3.3 in Grimmett–Stirzaker
2. (a) See p. 418 of Grimmett–Stirzaker
   (b) See Lemma 10.2.9 of Grimmett–Stirzaker
   (c) See p. 418 of Grimmett–Stirzaker
   (d) For example $T = \max\{k \geq 1 : X_1, \ldots, X_k < \frac{1}{2}\mathbb{E}[X_1]\}$
3. (a) We have that
$$J_n = X_1 + \min\{0, X_2, X_2 + X_3, \ldots, X_2 + X_3 + \cdots + X_n\} = X_1 + \min\{0, J'_{n-1}\}$$
   where J'_{n-1} has the same distribution as J_{n-1}. Thus $\mathbb{E}[J_n] = \mu + \mathbb{E}[\min\{0, J'_{n-1}\}] = \mu + \mathbb{E}[\min\{0, J_{n-1}\}]$.
   (b) Using $\min\{0, J_{n-1}\} - J_{n-1} = -\max\{0, J_{n-1}\} = J^+_{n-1}$ we get from (a) that $\mathbb{E}[J_n - J_{n-1}] = \mu - \mathbb{E}[J^+_{n-1}]$. Summing these from 2 to $n+2$ gives
$$\sum_{k=1}^{n} \mathbb{E}[J_k^+] = n\mu + \mathbb{E}[J_1 - J_{n+1}] \geq n\mu, \qquad \text{for } n \geq 2.$$
   where we used that $J_1 \geq J_2 \geq \cdots \geq J_{n+1}$. Then for any $m \geq 2$,
$$\mu \leq \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}[J_k^+] = \frac{1}{n}\sum_{k=1}^{m-1} \mathbb{E}[J_k^+] + \frac{1}{n}\sum_{k=m}^{n} \mathbb{E}[J_k^+] \leq \frac{1}{n}\sum_{k=1}^{m-1} \mathbb{E}[J_k^+] + \frac{n-m+1}{n}\mathbb{E}[J_m^+].$$
   where we used that $J_1^+ \geq J_2^+ \geq \cdots \geq J_{n+1}^+$. Letting $n \to \infty$ gives $\mathbb{E}[J_m^+] \geq \mu$ for any $m \geq 2$.
   (c) We have that $J_n \to \inf_{n\geq 1} S_n =: J_\infty$ almost surely and the limit is decreasing. Then $\mathbb{E}[J_\infty^+] \geq \mu > 0$ by the previous part and monotone convergence. But then $\mathbb{P}(J_\infty > 0) > 0$ which implies $\mathbb{P}(J_\infty > -\infty) > 0$. The event $\{J_\infty > -\infty\}$ is shift-invariant so then $\mathbb{P}(J_\infty > -\infty) = 1$.
   (d) We get that $\mathbb{P}(\liminf_{n\to\infty} S_n/n \geq 0) = 1$ for any X_n as in the statement. Applying this to $\tilde{X}_n = X_n - c$, where $c < \mu$, gives $\mathbb{P}(\liminf_{n\to\infty} S_n/n \geq c) = 1$ and then $\mathbb{P}(\liminf_{n\to\infty} S_n/n \geq \mu) = 1$. Applying it then to $-X_n$ gives the reverse inequality $\mathbb{P}(\limsup_{n\to\infty} S_n/n \leq \mu) = 1$ also.
4. (a) The renewal equation for $t \in [0,1]$ is
$$m(t) = F(t) + \int_0^t m(t-u)\, dF(u) = t + \int_0^t m(u)\, du$$
   where we used a change of variables. Then $m'(t) = 1 + m(t)$ with $m(0) = 0$ which gives $m(t) = e^t - 1$ as claimed.

(b) We have $N(t) = \mathbb{1}\{X_1 \leq t\}(1+\tilde{N}(t-X_1))$ where $\tilde{N}(t)$ is a copy of $N(t)$ independent of X_1. Then

$$N(t)^2 = \mathbb{1}\{X_1 \leq t\}(1 + 2\tilde{N}(t - X_1) + \tilde{N}(t - X_1)^2).$$

which gives (below we always assume $t \in [0, 1]$)

$$m_2(t) := \mathbb{E}[N(t)^2] = H(t) + \int_0^t m_2(t-u)\, dF(u)$$

where

$$H(t) = F(t) + 2\int_0^t m(t-u)\, dF(u) = 2m(t) - F(t).$$

Using Thm 10.1.11 this gives

$$m_2(t) = H(t) + \int_0^t H(t-u)\, dm(u) = 1 - e^t + 2te^t.$$

Here we used that $dm(u) = m'(u)du = e^u du$ and performed the integral. Then

$$\mathrm{Var}(N(t)) = m_2(t) - m(t)^2 = 2te^t - e^{2t} + e^t.$$

5. (a) See the book.

(b) By induction Z_{n-1} is $\mathcal{F}_{n-2}$-measurable, and the other terms $\mathbb{E}[X_n \mid \mathcal{F}_{n-1}]$ and X_{n-1} are $\mathcal{F}_{n-1}$-measurable, so Z_n is $\mathcal{F}_{n-1}$-measurable as claimed. We have $Z_n \geq Z_{n-1}$ since X_n is a submartingale. Further, Z_n is integrable for each n by induction. Defining $M_n = X_n - Z_n$ we readily see that M_n is a martingale, and $X_n = M_n + Z_n$ is the required decomposition.

(c) By the conditional Jensen inequality: $\mathbb{E}[Y_n^2 \mid \mathcal{F}_{n-1}] \geq \mathbb{E}[Y_n \mid \mathcal{F}_{n-1}]^2 = Y_{n-1}^2$. It readily follows that $(X_n)_{n\geq 0}$ is a submartingale. We have that

$$Z_n = \sum_{m=1}^{n} \left(\mathbb{E}[Y_m^2 \mid \mathcal{F}_{m-1}] - Y_{m-1}^2\right) = \sum_{m=1}^{n} \mathbb{E}[Y_m^2 - Y_{m-1}^2 \mid \mathcal{F}_{m-1}].$$

Note that $Y_m Y_{m-1}$ is integrable (by Cauchy–Schwarz) and $\mathbb{E}[Y_m Y_{m-1} \mid \mathcal{F}_{m-1}] = Y_{m-1}\mathbb{E}[Y_m \mid \mathcal{F}_{m-1}] = Y_{m-1}^2$ so that

$$\mathbb{E}[(Y_m - Y_{m-1})^2 \mid \mathcal{F}_{m-1}] = \mathbb{E}[Y_m^2 + Y_{m-1}^2 - 2Y_m Y_{m-1} \mid \mathcal{F}_{m-1}] = \mathbb{E}[Y_m^2 - Y_{m-1}^2 \mid \mathcal{F}_{m-1}].$$

The claim follows.

(d) The a.s. limit Z_∞ exists since $Z_n \geq Z_{n-1}$ for each $n \geq 1$. Next, we use Doob's L^r-inequality: if $r > 1$ and X_n is a non-negative submartingale such that $\mathbb{E}[X_n^r] < \infty$ for all $n \geq 0$ then $\mathbb{E}[\max_{0\leq m\leq n} X_m^r] \leq \left(\frac{r}{r-1}\right)^r \mathbb{E}[X_n^r]$. We can use this with $r = 2$ and $X_n = |Y_n|$ which is a submartingale by the same argument as for Y_n^2. We get that $\mathbb{E}[\max_{0\leq m\leq n} Y_m^2] \leq 4\mathbb{E}[Y_n^2]$. Next, $Y_n^2 = M_n + Z_n$ as above, where M_n is a martingale satisfying $M_0 = 0$. Then $\mathbb{E}[Y_n^2] = \mathbb{E}[Z_n]$. Also, Z_n is non-negative and increasing so using the monotone convergence theorem:

$$\mathbb{E}\big[\sup_{n\geq 0} Y_n^2\big] = \lim_{n\to\infty} \mathbb{E}\big[\max_{0\leq m\leq n} Y_m^2\big] \leq 4 \lim_{n\to\infty} \mathbb{E}[Z_n] = 4\mathbb{E}[Z_\infty].$$