EXAMINATION 2022-06-03 Examiner:
Stochastic Processes MVE330/MSF200 J. Björnberg

This examination consists of 7 questions for a total maximum of 50 points. Any bonus points are added on top, and grading is as follows. CTH: 20, 30, 40 points for 3, 4, 5 respectively. GU: 20, 35 points for G, VG respectively. PhD-students: 20 points for pass.

No tools allowed (pen is OK). **Good luck!**

1. (a) State the two Borel–Cantelli lemmas for a sequence of events $A_1, A_2, \dots$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. (2p)

   Randomly sample a number X in $[0,1]$ as follows: let $X_1, X_2, \dots$ be independent random variables uniformly distributed in $\{0,1,\dots,9\}$ and let X be the number such that X_n is the n:th digit in the decimal expansion of X.

   (b) Show that the pattern 123456789 appears infinitely often in the decimal expansion of X, with probability 1. (2p)

   (c) Let Y_n be the length of the 'run' of 0's starting from position n, that is $Y_n = \sup\{k \geq 1 : X_i = 0 \text{ for all } n \leq i \leq n+k-1\}$ (where $\sup \varnothing = 0$). Show that

   $$\mathbb{P}\big(\limsup_{n\to\infty} \tfrac{Y_n}{\log_{10} n} \leq 1\big) = 1. \qquad (3p)$$

2. Recall that a stochastic process $(Y_n)_{n\geq 0}$ is *uniformly integrable* if

   $$\sup_{n\geq 0} \mathbb{E}\big[|Y_n| \mathbb{1}\{|Y_n| \geq a\}\big] \to 0, \qquad \text{as } a \to \infty.$$

   (a) Show that a process $(Y_n)_{n\geq 0}$ of the form $Y_n = \mathbb{E}[Y \mid \mathcal{F}_n]$, for $(\mathcal{F}_n)_{n\geq 0}$ a sequence of σ-algebras and Y an integrable random variable, is uniformly integrable. (2p)

   (b) Show that a process $(Y_n)_{n\geq 0}$, such that $\sup_{n\geq 0} \mathbb{E}[|Y_n|^r] < \infty$ for some $r > 1$, is uniformly integrable. (2p)

   (c) Assuming that a stochastic process $(Y_n)_{n\geq 0}$ is uniformly integrable, and that $Y_n \to Y$ almost surely, prove that $\mathbb{E}|Y| < \infty$. (2p)

3. (a) Define what is means for $(N(t) : t \geq 0)$ to be a *renewal process.* (2p)

   (b) State and prove the renewal equation for the function $m(t) = \mathbb{E}[N(t)]$. (3p)

   (c) Trams stop at Järntorget on average twice per minute, and for tram i a random number Y_i of passengers get off while Z_i get on. Making reasonable assumptions, and stating clearly any standard results you use, find an expression for the long-term average number of passengers using the tram stop. (2p)

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4. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(Y_n)_{n\geq 0}$ a martingale.

   (a) Define a *stopping time* T as well as the σ-algebra $\mathcal{F}_T$. (2p)

   (b) Prove that for each n the random variable $Y_{T\wedge n}$ is integrable, i.e. $\mathbb{E}|Y_{T\wedge n}| < \infty$. (2p)

   (c) Assuming that $\mathbb{P}(T < \infty) = 1$, prove that $\mathbb{E}[Y_n \mid \mathcal{F}_T] = Y_{T\wedge n}$. (2p)

5. In this problem, you may assume the results of the previous problem, as well as the following: *If $(Y_n)_{n\geq 0}$ is a martingale and T a stopping time, then $(Y_{T\wedge n})_{n\geq 0}$ is a martingale.*

   Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{P}(X_i = +1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$ and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Also let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$.

   (a) For $\theta > 0$, show that $Y_n = \frac{\exp(\theta S_n)}{\cosh(\theta)^n}$ is a martingale. (2p)

   (b) For $k \geq 0$, let $T_k = \inf\{n \geq 0 : S_n = k\}$. Carefully prove that $\mathbb{P}(T_k < \infty) = 1$ and establish the following formula, valid for $\theta > 0$:
   $$\mathbb{E}[\cosh(\theta)^{-T_k}] = e^{-\theta k}. \qquad (3p)$$

   (c) Show that $\mathbb{E}[Y_{T_k} \mid \mathcal{F}_{T_{k-1}}] = Y_{T_{k-1}}$. (2p)

   (d) Deduce that the differences $(T_k - T_{k-1})_{k\geq 1}$ are independent copies of T_1. (2p)

6. This problem concerns *Pólya's urn*: starting with one blue and one red ball in an urn, we repeatedly (and independently) pick a ball from the urn uniformly at random and replace it with *two* balls of the same colour. Let R_n denote the number of red balls after n steps, and $Y_n = R_n/(n+2)$ the *fraction* of red balls. Consider the filtration given by $\mathcal{F}_n = \sigma(R_0, R_1, \ldots, R_n)$.

   (a) Show that Y_n is a martingale with respect to this filtration, which converges almost surely and in L^1 to some random variable Y_∞. (2p)

   (b) Show that, with probability at least $\frac{2}{3}$, the proportion of red balls stays below $\frac{3}{4}$ forever. (2p)

   (c) Calculate the probability of the first n draws consisting of m red balls followed by $n - m$ blue balls. (2p)

   (d) Calculate the probability $\mathbb{P}(R_n = m + 1)$. Use your expression to show that Y_∞ is uniformly distributed. (2p)

7. (a) Define what it means for $(B(t))_{t\geq 0}$ to be a *standard Brownian motion.* (2p)

   (b) If $(B(t))_{t\geq 0}$ is a standard Brownian motion and $a > 0$ is a constant, show that $(\frac{1}{a}B(a^2 t))_{t\geq 0}$ is a standard Brownian motion. (2p)

   (c) If $(B(t))_{t\geq 0}$ is a standard Brownian motion, prove that, with probability 1, there is no interval $[a, b]$ with $a < b$ on which $B(t)$ is monotone (i.e. either increasing or decreasing). (3p)