1. (a) *The first Borel–Cantelli Lemma:* If $\sum_{k=1}^{\infty} \mathbb{P}(A_k) < \infty$ then $\mathbb{P}(A_k \text{ i.o.}) = 0$.
The second Borel–Cantelli Lemma: If the A_k are independent and $\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty$ then $\mathbb{P}(A_k \text{ i.o.}) = 1$.

   (b) Let A_k be the event that the pattern 123456789 occurs starting from position $9k$. Then the A_k are independent and $\mathbb{P}(A_k) = 10^{-9}$. So $\sum_{k=1}^{\infty} \mathbb{P}(A_k) = \infty$ and by BCII A_k happens infinitely often.

   (c) For any integer $r \geq 1$ we have $\mathbb{P}(Y_n \geq r) = 10^{-r}$. So for any $\varepsilon > 0$ we have
$$\mathbb{P}(Y_n \geq (1+\varepsilon)\log_{10} n) = 10^{-\lceil (1+\varepsilon)\log_{10} n \rceil} \leq n^{-(1+\varepsilon)}.$$
These are summable so by BCI we have $\mathbb{P}\big(Y_n \geq (1+\varepsilon)\log_{10} n \text{ i.o.}\big) = 0$. Then
$$\mathbb{P}\Big(\limsup_{n\to\infty} \tfrac{Y_n}{\log_{10} n} < 1+\varepsilon\Big) = 1$$
which gives the claim.

2. (a) By the conditional Jensen inequality, $|X_n| \leq \mathbb{E}[|Y| \mid \mathcal{F}_n] =: Z_n$. This implies that
$$\begin{aligned}\mathbb{E}[|X_n|\mathbb{1}\{|X_n| \geq a\}] &\leq \mathbb{E}[|Z_n|\mathbb{1}\{Z_n \geq a\}] = \mathbb{E}[|Y|\mathbb{1}\{Z_n \geq a\}]\\ &\leq \mathbb{E}[|Y|\mathbb{1}\{|Y| \geq \sqrt{a}\}] + \sqrt{a}\,\mathbb{P}(Z_n \geq a)\\ &\leq \mathbb{E}[|Y|\mathbb{1}\{|Y| \geq \sqrt{a}\}] + \sqrt{a}\frac{\mathbb{E}[Z_n]}{a}\\ &= \mathbb{E}[|Y|\mathbb{1}\{|Y| \geq \sqrt{a}\}] + \frac{\mathbb{E}[|Y|]}{\sqrt{a}}.\end{aligned}$$
The right side goes to zero as $a \to \infty$ since Y is integrable, and it does not depend on n.

   (b) Writing $M = \sup_{n\geq 0} \mathbb{E}[|Y_n|^r]$ we have
$$\mathbb{E}[|Y_n|\mathbb{1}\{|Y_n| \geq a\}] = \mathbb{E}[|Y_n|^r \tfrac{1}{|Y_n|^{r-1}}\mathbb{1}\{|Y_n| \geq a\}] \leq \tfrac{1}{a^{r-1}}\mathbb{E}[|Y_n|^r] \leq \frac{M}{a^{r-1}} \to 0, \qquad \text{as } a \to \infty.$$

   (c) By Fatou's Lemma,
$$\begin{aligned}\mathbb{E}|Y| &= \mathbb{E}[\lim_{n\to\infty} |Y_n|] \leq \liminf_{n\to\infty} \mathbb{E}|Y_n| = \liminf_{n\to\infty} \big(\mathbb{E}[|Y_n|\mathbb{1}\{|Y_n| \geq a\}] + \mathbb{E}[|Y_n|\mathbb{1}\{|Y_n| < a\}]\big)\\ &\leq a + \sup_{n\geq 0} \mathbb{E}[|Y_n|\mathbb{1}\{|Y_n| \geq a\}].\end{aligned}$$
For large enough $a > 0$, the right-hand-side is at most $a+1$.

3. (a) *Renewal process:* $N(t) = \#\{k \geq 1 : X_1 + X_2 + \cdots + X_k \leq t\}$ where $X_1, X_2, \ldots$ are i.i.d. non-negative random variables.

   (b) *The renewal equation:* If the X_i are integrable and $\mathbb{P}(X_i > 0) = 1$, then with $F(t) = \mathbb{P}(X_i \leq t)$,
$$m(t) = F(t) + \int_0^t m(t-s)\,dF(s).$$
Proof: Let $\tilde{N}(t) = \#\{k \geq 2 : X_2 + \cdots + X_k \leq t\}$. Then
$$\begin{aligned}m(t) &= \mathbb{E}[N(t)\mathbb{1}\{X_1 > t\}] + \mathbb{E}[N(t)\mathbb{1}\{X_1 \leq t\}] = 0 + \mathbb{E}[(1 + \tilde{N}(t - X_1))\mathbb{1}\{X_1 \leq t\}]\\ &= F(t) + \mathbb{E}\big[\mathbb{E}[\tilde{N}(t-X_1) \mid X_1]\mathbb{1}\{X_1 \leq t\}\big] = F(t) + \mathbb{E}[m(t-X_1)\mathbb{1}\{X_1 \leq t\}],\end{aligned}$$
as claimed. We used that, given X_1, the process $\tilde{N}(t - X_1)$ is an independent copy of $N(t)$, so that $\mathbb{E}[\tilde{N}(t-X_1) \mid X_1]\mathbb{1}\{X_1 \leq t\} = m(t-X_1)\mathbb{1}\{X_1 \leq t\}$.

(c) Assume that the triples (X_i, Y_i, Z_i) are i.i.d. where X_i are the inter-tram times. Let $C_i = Y_i + Z_i$ be the number of users of the i:th tram. Then $C(t) = \sum_{i=1}^{N(t)} C_i$ is a renewal-reward process, so by the renewal-reward theorem

$$\frac{C(t)}{t} \to \frac{\mathbb{E}[C_1]}{\mathbb{E}[X_1]} = 2(\mathbb{E}[Y_1] + \mathbb{E}[Z_1])$$

almost surely and in mean.

4. (a) *Stopping time:* random variable T with values in $\{0, 1, 2, \dots\} \cup \{\infty\}$ such that for each $n \geq 0$ we have $\{T \leq n\} \in \mathcal{F}_n$.
The σ-algebra $\mathcal{F}_T$: all $A \in \mathcal{F}$ such that $A \cap \{T = n\} \in \mathcal{F}_n$ for all $n \geq 0$.

(b) Since

$$Y_{T\wedge n} = \sum_{k=0}^{n-1} Y_k \mathbb{1}\{T = k\} + Y_n \mathbb{1}\{T \geq n\}$$

we have

$$|Y_{T\wedge n}| \leq \sum_{k=0}^{n} |Y_k|,$$

all terms of which have finite expectation since this is part of being a martingale.

(c) Let $A \in \mathcal{F}_T$ and write $W = \mathbb{E}[Y_n \mid \mathcal{F}_T]$. Then

$$\begin{aligned}
\mathbb{E}[W \mathbb{1}_A] = \mathbb{E}[Y_n \mathbb{1}_A] &= \sum_{k=0}^{n-1} \mathbb{E}[Y_n \mathbb{1}_{A\cap\{T=k\}}] + \mathbb{E}[Y_n \mathbb{1}_{A\cap\{T\geq n\}}] \\
&= \sum_{k=0}^{n-1} \mathbb{E}\big[\mathbb{E}[Y_n \mid \mathcal{F}_k] \mathbb{1}_{A\cap\{T=k\}}\big] + \mathbb{E}[Y_n \mathbb{1}_{A\cap\{T\geq n\}}] \\
&= \sum_{k=0}^{n-1} \mathbb{E}\big[Y_{k\wedge n} \mathbb{1}_{A\cap\{T=k\}}\big] + \mathbb{E}[Y_n \mathbb{1}_{A\cap\{T\geq n\}}] \\
&= \mathbb{E}[Y_{T\wedge n} \mathbb{1}_A],
\end{aligned}$$

as required. Here we used the martingale property which means that $\mathbb{E}[Y_n \mid \mathcal{F}_k] = Y_{k\wedge n}$. (Note that we did not need to use $\mathbb{P}(T < \infty) = 1$ in fact.)

5. (a) Clearly it is adapted and integrable. We have

$$\mathbb{E}\Big[\tfrac{e^{\theta S_n}}{\cosh(\theta)^n} \mid \mathcal{F}_{n-1}\Big] = \tfrac{e^{\theta S_{n-1}}}{\cosh(\theta)^{n-1}} \mathbb{E}\Big[\tfrac{e^{\theta X_n}}{\cosh(\theta)} \mid \mathcal{F}_{n-1}\Big].$$

But

$$\mathbb{E}\Big[\tfrac{e^{\theta X_n}}{\cosh(\theta)} \mid \mathcal{F}_{n-1}\Big] = \mathbb{E}\Big[\tfrac{e^{\theta X_n}}{\cosh(\theta)}\Big] = 1$$

by independence and explicit computation.

(b) The process $Y_{T_k\wedge n}$ is a martingale, thus $\mathbb{E}[Y_{T_k\wedge n}] = Y_0 = 1$. As $n \to \infty$, we have that $Y_{T_k\wedge n} \to Y_{T_k} \mathbb{1}\{T_k < \infty\}$ a.s. Moreover, since $\theta > 0$ we have $Y_{T_k\wedge n} \leq e^{-k\theta}$ so by the bounded convergence theorem we get $\mathbb{E}[Y_{T_k} \mathbb{1}\{T_k < \infty\}] = 1$, that is

$$\mathbb{E}\Big[\tfrac{e^{k\theta}}{\cosh(\theta)^{T_k}} \mathbb{1}\{T_k < \infty\}\Big] = 1, \qquad \text{or} \qquad \mathbb{E}\big[\cosh(\theta)^{-T_k} \mathbb{1}\{T_k < \infty\}\big] = e^{-k\theta}.$$

Letting $\theta \downarrow 0$ and applying the monotone convergence theorem we get $\mathbb{P}(T_k < \infty) = 1$, as required. The formula also follows.

(c) Since $Y_{T_k\wedge n}$ is a martingale and we now know that $\mathbb{P}(T_{k-1}<\infty)=1$, the previous problem tells us that $\mathbb{E}[Y_{T_k\wedge n}\mid \mathcal{F}_{T_{k-1}}]=Y_{T_k\wedge n\wedge T_{k-1}}=Y_{T_{k-1}\wedge n}$. We used that $T_{k-1}\le T_k$. Using that $\mathbb{P}(T_{k-1}<\infty)=1$ we see that $Y_{T_{k-1}\wedge n}\to Y_{T_{k-1}}$ a.s., thus $\mathbb{E}[Y_{T_k\wedge n}\mid \mathcal{F}_{T_{k-1}}]\to Y_{T_{k-1}}$ a.s. To conclude, take $A\in\mathcal{F}_{T_{k-1}}$ and note that

$$\mathbb{E}\big[\mathbb{E}[Y_{T_k\wedge n}\mid \mathcal{F}_{T_{k-1}}]\mathbb{1}_A\big]=\mathbb{E}[Y_{T_k\wedge n}\mathbb{1}_A].$$

The right side converges to $\mathbb{E}[Y_{T_k}\mathbb{1}_A]$ due to a.s. convergence and the bounded convergence theorem. The left side converges to $\mathbb{E}[Y_{T_{k-1}}\mathbb{1}_A]$ by what was shown above (and the bounded convergence theorem). Thus $\mathbb{E}[Y_{T_k}\mid\mathcal{F}_{T_{k-1}}]=Y_{T_{k-1}}$ as claimed.

(d) The previous part says

$$\tfrac{e^{\theta(k-1)}}{\cosh(\theta)^{T_{k-1}}}=\mathbb{E}\big[\tfrac{e^{\theta k}}{\cosh(\theta)^{T_k}}\mid\mathcal{F}_{T_{k-1}}\big]=\mathbb{E}\big[\tfrac{e^{\theta k}}{\cosh(\theta)^{T_k-T_{k-1}}}\mid\mathcal{F}_{T_{k-1}}\big]\tfrac{1}{\cosh(\theta)^{T_{k-1}}}$$

where the factor we took out was measurable and bounded. Performing cancellations and taking expectation gives

$$\mathbb{E}\big[\tfrac{1}{\cosh(\theta)^{T_k-T_{k-1}}}\big]=e^{-\theta}.$$

Since

$$(e^{-\theta})^k=\mathbb{E}\big[\tfrac{1}{\cosh(\theta)^{T_k}}\big]=\mathbb{E}\big[\prod_{i=0}^{k}\tfrac{1}{\cosh(\theta)^{T_i-T_{i-1}}}\big]$$

and since the probability generating function determines the distribution, we conclude the claim.

6. (a) We have

$$\mathbb{E}[R_{n+1}\mid\mathcal{F}_n]=R_n\tfrac{n+2-R_n}{n+2}+(R_n+1)\tfrac{R_n}{n+2}=\tfrac{(n+3)R_n}{n+2}.$$

Thus

$$\mathbb{E}[Y_n\mid\mathcal{F}_n]=\mathbb{E}\big[\tfrac{R_{n+1}}{n+3}\mid\mathcal{F}_n\big]=\tfrac{R_n}{n+2}=Y_n$$

as required. Also, $0\le Y_n\le 1$ and a bounded process is UI. So by the UI martingale convergence theorem, $Y_n\to Y_\infty$ a.s. and in L^1 for some Y_∞.

(b) We use the *maximal inequality:* if $(Y_n)_{n\ge0}$ is a submartingale then for any $x>0$ we have

$$\mathbb{P}\big(\max_{1\le k\le n}Y_k\ge x\big)\le\frac{1}{x}\mathbb{E}[Y_n^+].$$

By the maximal inequality, and $\mathbb{E}[Y_n^+]=\mathbb{E}[Y_n]=Y_0=\frac12$,

$$\mathbb{P}\big(\sup_{n\ge1}Y_n\ge\tfrac34\big)=\lim_{n\to\infty}\mathbb{P}\big(\max_{1\le k\le n}Y_k\ge\tfrac34\big)\le\tfrac23.$$

(c) The probability of drawing m red balls followed by $n-m$ blue balls is

$$\frac{1}{2}\frac{2}{3}\cdots\frac{m}{m+1}\cdot\frac{1}{m+2}\frac{2}{m+3}\cdots\frac{n-m}{n+1}=\frac{m!(n-m)!}{(n+1)!}.$$

(d) The probability of picking m red balls and $n-m$ blue balls, in some other fixed order, is obtained by permuting the factors in the numerator of the previous expression. That is, all such sequences of picks have the same probability. There are $\binom{n}{m}$ such sequences, thus

$$\mathbb{P}(R_n=m+1)=\binom{n}{m}\frac{m!(n-m)!}{(n+1)!}=\frac{1}{n+1},\qquad 0\le m\le n.$$

Then for any $0 \leq t \leq 1$,

$$\mathbb{P}(Y_n \leq t) = \mathbb{P}(R_n \leq \lfloor nt \rfloor) = \sum_{m=0}^{\lfloor nt \rfloor - 1} \mathbb{P}(R_n = m+1) = \frac{\lfloor nt \rfloor}{n+1} \to t.$$

So Y_n converges in distribution to a uniform random variable. Since we know that $Y_n \to Y_\infty$ a.s., the limit Y_∞ must be uniformly distributed.

7. (a) Starts at 0, increments independent and law $B(t+s) - B(t) \sim N(0, s)$, and continuous with probability 1.

   (b) All properties are immediate except for the distribution of the increments. But $B(a^2 t + a^2 s) - B(a^2 s)$ is normal with mean 0 and variance $a^2 s$, so dividing it by a gives the correct variance s.

   (c) By taking a union over $a, b \in \mathbb{Q}$, suffices to consider fixed a, b and the event of being increasing on $[a, b]$. We can subdivide $[a, b]$ into n non-empty intervals on each of which the increment is normally distributed with mean 0. Then the probability of each of these increments being ≥ 0 is $2^{-n} \to 0$.