TMA970 Inledande matematisk analys F1 / TM1

Lösningar 26/10-22

1. (a) konvergent ; (b) konvergent ;
(c) konvergent ; (d) falskt ;
(e) sant ; (f) sant.

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2. (a) Sätt $t = \frac{\pi}{2} - x \iff x = \frac{\pi}{2} - t$

$x \to \frac{\pi}{2} \iff t \to 0$

$$\frac{\sqrt{1+\sin x} - \frac{2\sqrt{2}}{\pi}x}{x - \frac{\pi}{2}} = -\frac{\sqrt{1+\cos t} - \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{2} - t\right)}{t} =$$

$$= -\frac{\sqrt{2\cos^2\frac{t}{2}} - \sqrt{2} + \frac{2\sqrt{2}}{\pi}t}{t} \underset{\uparrow}{=} -\frac{2\sqrt{2}}{\pi} - \sqrt{2}\,\frac{\cos\frac{t}{2} - 1}{t} =$$

$\cos\frac{t}{2} > 0$ nära 0

$\Rightarrow \left|\cos\frac{t}{2}\right| = \cos\frac{t}{2}$

$$= -\frac{2\sqrt{2}}{\pi} + \sqrt{2}\cdot\frac{\left(1-\cos\frac{t}{2}\right)\left(1+\cos\frac{t}{2}\right)}{t\left(1+\cos\frac{t}{2}\right)} =$$

$$= -\frac{2\sqrt{2}}{\pi} + \sqrt{2}\cdot\frac{\sin^2\frac{t}{2}}{4\left(\frac{t}{2}\right)^2}\cdot t\cdot\frac{1}{1+\cos\frac{t}{2}} \xrightarrow[\substack{t\to 0\\ x \to \frac{\pi}{2}}]{} \underline{\underline{-\frac{2\sqrt{2}}{\pi}}}$$

(b) $$\frac{x^2 + \sin x}{x\ln x + \cos x} = \underbrace{\frac{x^{\cancel{2}}}{\cancel{x}\ln x}}_{\to\,\infty} \cdot \underbrace{\frac{1 + \overbrace{\frac{\sin x}{x^2}}^{\to 0}}{1 + \underbrace{\frac{\cos x}{x\ln x}}_{\to 0}}}_{\to\,1 \;\; x\to\infty} \xrightarrow[x\to\infty]{} \infty$$

③. $f(x) = \sqrt[3]{x^2} - \sqrt[3]{x^2+1}$ (2)

$D_f = \mathbb{R}$

$\sqrt[3]{x^2+1} > \sqrt[3]{x^2} \quad \forall x \in \mathbb{R}$

$\Rightarrow f(x) > 0 \quad \forall x \in \mathbb{R}$

Symmetrier: $f(-x) = f(x) \Rightarrow$ jämn

ej periodisk

Inga vertikala asymptoter.

$$\sqrt[3]{x^2} - \sqrt[3]{x^2+1} = \frac{x^2 - (x^2+1)}{\left(\sqrt[3]{x^2}\right)^2 + \sqrt[3]{x^2}\sqrt[3]{x^2+1} + \left(\sqrt[3]{x^2+1}\right)^2} =$$

$$= \frac{-1}{x^{4/3} + x^{2/3}(x^2+1)^{1/3} + (x^2+1)^{2/3}} \xrightarrow[x \to \pm\infty]{} 0$$

$\rightarrow$ horisontell asymptot i $\pm\infty$

$y = 0$

(räcker i $+\infty$, eftersom f är jämn)

$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} + \frac{(-1)}{3}(x^2+1)^{-\frac{2}{3}} \cdot 2x =$$

$D_{f'}$: $\boxed{x \neq 0}$

$$= \frac{2\left((x^2+1)^{\frac{2}{3}} - x^{\frac{4}{3}}\right)}{3x^{\frac{1}{3}}(x^2+1)^{\frac{2}{3}}}$$

täljaren $> 0 \quad \forall x \in D_{f'}$

nämnaren $\begin{cases} < 0 & \forall x < 0 \\ > 0 & \forall x > 0 \end{cases}$

$\Rightarrow f$ avtagande i $(-\infty, 0)$

växande i $(0, +\infty)$ $\quad f(0) = -1$

$\Rightarrow f$ har lokalt (och globalt) minimum i 0

$$f''(x) = -\frac{2}{9}x^{-\frac{4}{3}} - \frac{2}{3}(x^2+1)^{-\frac{2}{3}} +$$

$$+\left(\frac{2}{3}\right)^2 x\,(x^2+1)^{-\frac{5}{3}} \cdot 2x =$$

$$= 2 \cdot \frac{-(x^2+1)^{\frac{5}{3}} - 3(x^2+1)x^{\frac{4}{3}} + 2^2 \cdot x^{\frac{10}{3}}}{9x^{\frac{4}{3}}\,(x^2+1)^{5/3}} =$$

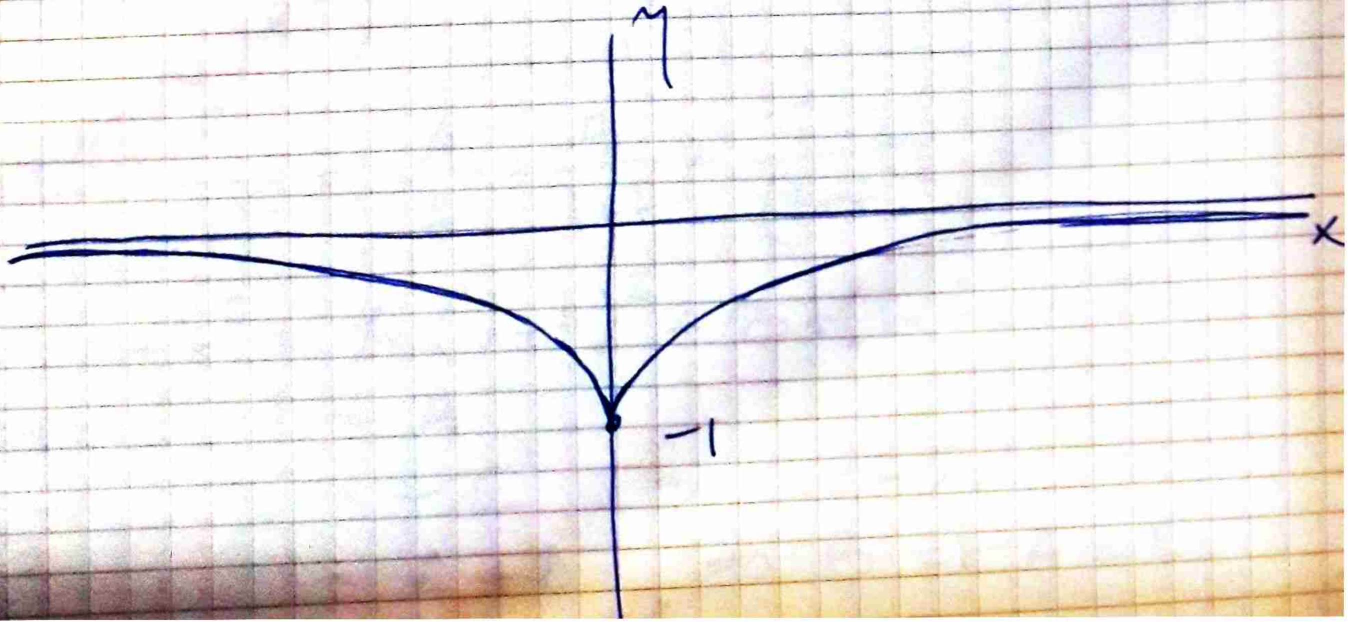
$$= -2\,\frac{(x^2+1)^{5/3} + 3x^{4/3} - x^{10/3}}{9x^{4/3}(x^2+1)^{5/3}} < 0, \quad \forall x \neq 0$$

eftersom $(x^2+1)^{5/3} > x^{10/3}$

$x^{4/3} > 0$

$\Rightarrow f$ konkav i $(-\infty, 0) \cup (0, \infty)$

x	$-\infty$			0			$+\infty$
f	$y=0$	$\searrow$		-1		$\nearrow$	
f'		$-$	$-$	$\to -\infty$ | $\to +\infty$	$+$	$+$	
f''	$-$	$-$	$-$	|	$-$	$-$	$-$



4. (a) $\int \cos^2\sqrt{x}\,dx = \begin{bmatrix} t=\sqrt{x} \\ x=t^2 \\ dx = 2t\,dt \end{bmatrix} =$

$$= \int 2t\cos^2 t\,dt = \int \cancel{2}t \cdot \frac{1+\cos 2t}{\cancel{2}}\,dt =$$

$$= \frac{t^2}{2} + \int t\cos 2t\,dt \overset{p.i.}{=}$$

$$= \frac{t^2}{2} + \frac{1}{2}t\sin 2t - \frac{1}{2}\int 1\cdot\sin 2t\,dt =$$

$$= \frac{t^2}{2} + \frac{1}{2}t\sin 2t + \frac{1}{2}\cdot\frac{1}{2}\cos 2t \quad (+C) =$$

$$= \frac{x}{2} + \frac{1}{2}\sqrt{x}\,\sin 2\sqrt{x} + \frac{1}{4}\cos 2\sqrt{x} \quad (+C)$$

(b) Sätt $e^{\frac{x}{4}} = t$, $x = 4\ln t$

$e^{\frac{x}{2}} = t^2$, $dx = \frac{4}{t}\,dt$

$x=0 : t=1$; $x=4 : t=e$

$$\int_0^4 \frac{1+e^{x/2}}{(1+e^{x/4})^2}\,dx = \int_1^e \frac{1+t^2}{(1+t)^2}\cdot\frac{4}{t}\,dt = \circledast$$

$$\frac{1+t^2}{t(1+t)^2} = \frac{A}{t} + \frac{B}{1+t} + \frac{C}{(1+t)^2}$$

$$1+t^2 = A(1+t)^2 + Bt(1+t) + Ct$$

$t=0$: $1 = A$

$t=-1$: $2 = -C \Rightarrow C=-2$

t^2 : $1 = A+B \Rightarrow B = 1-A = 0$

$$\circledast = \left[\ln t + 2\cdot\frac{1}{1+t}\right]_1^e = \cancel{1} + \frac{2}{1+e} - 0 - \cancel{1} = \frac{2}{1+e}$$

5. $D_f = \mathbb{R}$ symmetrisk m.a.p. 0

$$f(-x) = \ln\left(-x + \sqrt{x^2+1}\right) =$$

$$= \ln \frac{\left(\sqrt{x^2+1} - x\right)\left(\sqrt{x^2+1} + x\right)}{\sqrt{x^2+1} + x} = \ln \frac{\cancel{x^2} + 1 - \cancel{x^2}}{\sqrt{x^2+1} + x} =$$

$$= -\ln\left(x + \sqrt{x^2+1}\right) = -f(x)$$

$\Rightarrow f$ udda

f deriverbar i $\mathbb{R}$, $f'(x) = \frac{1}{\sqrt{x^2+1}} > 0$

(tabellderivata)

$\Rightarrow f$ strängt växande i $\mathbb{R}$

$\Rightarrow f$ injektiv $\Rightarrow f : \mathbb{R} \to V_f$ bijektion

V_f: $\lim_{x \to +\infty} f(x) = $ "$\ln(+\infty)$" $= \infty$

f udda $\Rightarrow \lim_{x \to -\infty} f(x) = -\infty$

$\Rightarrow V_f = \mathbb{R}$

$$\ln\left(x + \sqrt{x^2+1}\right) = y$$

$$\Leftrightarrow x + \sqrt{x^2+1} = e^y$$

$$\sqrt{x^2+1} = e^y - x \quad \Big|^2$$

$$\cancel{x^2} + 1 = e^{2y} - 2xe^y + \cancel{x^2}$$

$$x = \frac{e^{2y} - 1}{2e^y} = \frac{e^y - e^{-y}}{2} \quad (= \sinh y)$$

6. $f:[a,b] \to \mathbb{R}$, $f \geq 0$

$x_0 \in [a,b]$ s.a. $f(x_0) > 0$

Sätt $\varepsilon_0 = \frac{1}{2} f(x_0) > 0$

$\exists \delta_{\varepsilon_0} > 0 : \forall x \in [a,b]$, $|x - x_0| < \delta_{\varepsilon_0}$

$\delta = $

$$| f(x) - f(x_0) | < \varepsilon_0 = \frac{1}{2} f(x_0)$$

$$\Leftrightarrow \underbrace{f(x_0) - \frac{1}{2} f(x_0)}_{= \frac{1}{2} f(x_0) > 0} < f(x) < f(x_0) + \frac{1}{2} f(x_0)$$

$$\Rightarrow \int_a^b f(x)\, dx \geq \int_{[x_0 - \delta, x_0 + \delta] \cap [a,b]} f(x)\, dx >$$

$$> \delta \cdot \frac{1}{2} f(x_0) > 0.$$