# TMA970 Inledande matematisk analys F1 / TM1

# Lösningar 5/1-23

**1.** (a) konvergent; (b) konvergent; (c) divergent; (d) sant; (e) sant; (f) falskt.

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**2.** (a)

$$\frac{\ln(x^2+1)}{\ln(x^3+1)} = \frac{\ln\left(x^2\left(1+\frac{1}{x^2}\right)\right)}{\ln\left(x^3\left(1+\frac{1}{x^3}\right)\right)} =$$

$$= \frac{\ln x^2 + \ln\left(1+\frac{1}{x^2}\right)}{\ln x^3 + \ln\left(1+\frac{1}{x^3}\right)} = \frac{2\ln x + \ln\left(1+\frac{1}{x^2}\right)}{3\ln x + \ln\left(1+\frac{1}{x^3}\right)} =$$

$$= \frac{2+\left(\ln\left(1+\frac{1}{x^2}\right)/\ln x\right)}{3+\left(\ln\left(1+\frac{1}{x^3}\right)/\ln x\right)} \xrightarrow[x\to\infty]{} \underline{\underline{\frac{2}{3}}}$$

(b)

$$\frac{\sin mx}{\sin nx} = \begin{bmatrix} x = \pi - t \\ x \to \pi \Leftrightarrow t \to 0 \end{bmatrix}$$

$$= \frac{\sin m(\pi-t)}{\sin n(\pi-t)} = \frac{\overbrace{\sin m\pi}^{=0}\cos mt - \overbrace{\cos m\pi}^{=(-1)^m}\sin mt}{\underbrace{\sin n\pi}_{=0}\cos nt - \underbrace{\cos n\pi}_{=(-1)^n}\sin nt} =$$

$$= (-1)^{m-n}\frac{\sin mt}{\sin nt}\cdot\frac{nt}{nt}\cdot\frac{mt}{mt} =$$

$$= (-1)^{m-n}\underbrace{\frac{\sin mt}{mt}}_{\to 1,\ t\to 0}\cdot\underbrace{\frac{nt}{\sin nt}}_{\to 1,\ t\to 0}\cdot\frac{m\cancel{t}}{n\cancel{t}} \xrightarrow[t\to 0]{} \underline{\underline{(-1)^{m-n}\frac{m}{n}}}$$

3. $f(x) = \dfrac{2x-1}{(x-1)^2}$ (2)

$D_f: x \neq 1$

$f(x) = 0$ för $x = \frac{1}{2}$ | $f > 0$ för $x > \frac{1}{2}$, $x \neq 1$

$f < 0$ för $x < \frac{1}{2}$

inga symmetrier

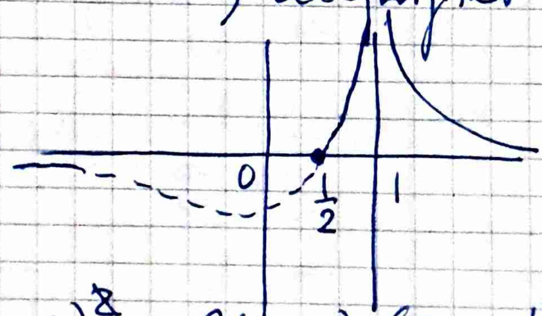
$\lim_{x \to \pm\infty} f(x) = 0 \quad \left(\frac{\text{grad } 1}{\text{grad } 2}\right)$

$\lim_{x \to 1\pm} \dfrac{2x-1}{(x-1)^2} = "\dfrac{1}{+0}" = +\infty$

$\Rightarrow$ vertikal asymptot $x = 1$

sned (horisontell) asymptot $y = 0$ i $\pm\infty$

Skiss:




$$f'(x) = \frac{2(x-1)^{\cancel{2}} - 2\cancel{(x-1)}(2x-1)}{(x-1)^{\cancel{4}\,3}} =$$

$$= \frac{2x - \cancel{2} - 4x + \cancel{2}}{(x-1)^3} = \frac{-2x}{(x-1)^3}$$

$f'(0) = 0$, $f' < 0 \ \forall x > 1$

$f' > 0 \ \forall x \in (0, 1)$

$f' < 0 \ \forall x < 0$

$\Rightarrow$ $f$ avtagande i $(-\infty, 0)$

$f$ har lok. min i $0$

$f$ växande i $(0, 1)$

$f$ avtagande i $(1, \infty)$

$$f''(x) = \frac{-2(x-1)^{\cancel{3}} - (-2x)\cdot 3\cancel{(x-1)^2}}{(x-1)^{\cancel{6}\,4}} =$$

$$= \frac{-2x+2+6x}{(x-1)^4} = 2\cdot\frac{2x+1}{(x-1)^4}$$

$f''=0$ i $x=-\frac{1}{2}$

$f''>0 \ \forall x>-\frac{1}{2},\ x\neq 1$ ; $f''<0 \ \forall x<-\frac{1}{2}$

$\Rightarrow f$ har inflexion i $-\frac{1}{2}$

konvex i $(-\frac{1}{2},1)$ och i $(1,\infty)$

konkav i $(-\infty,-\frac{1}{2})$

| $x$ | $-\infty$ | | $-\frac{1}{2}$ | | $0$ | | $\frac{1}{2}$ | | $1$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f$ | $y=0$ as. | $\searrow$ | infl. | $\searrow$ | $-1$ lok. min | $\nearrow$ | $0$ | $\nearrow$ | $+\infty$ \| $+\infty$ | $\searrow$ | $y=0$ as. |
| $f'$ | | $-$ | $-$ | | $0$ | $+$ | | | \| | $-$ | $-$ |
| $f''$ | | $-$ | $-$ | $0$ | | $+$ | | | \| | $+$ | $+$ |

4. (a) $\int x^2 \ln^2 x \, dx \overset{p.i.}{=}$

$$= \frac{x^3}{3}\ln^2 x - \int \frac{x^{\cancel{3}2}}{3}\cdot 2\ln x \cdot \frac{1}{\cancel{x}}\,dx \overset{p.i.}{=}$$

$$= \frac{x^3}{3}\ln^2 x - \frac{2x^3}{9}\ln x + \int \frac{2x^{\cancel{3}2}}{9}\cdot\frac{1}{\cancel{x}}\,dx$$

$$= \frac{x^3}{3}\ln^2 x - \frac{2x^3}{9}\ln x + \frac{2x^3}{27} \quad (+C)$$

(b) standardsubstitutionen:

$$t = \tan\frac{x}{2} \qquad x = 2\arctan t$$

$$dx = \frac{2}{1+t^2}\,dt$$

$$\cos x = \frac{1-t^2}{1+t^2} \qquad \sin x = \frac{2t}{1+t^2}$$

$$\int \frac{1}{2\sin x - \cos x + 5}\,dx = \int \frac{2/(1+t^2)}{\frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2} + 5}\,dt =$$

$$= \int \frac{2}{4t - 1 + t^2 + 5 + 5t^2}\,dt = \int \frac{dt}{3t^2 + 2t + 2} =$$

$$= \frac{1}{3}\int \frac{dt}{t^2 + \frac{2}{3}t + \frac{2}{3}} = \frac{1}{3}\int \frac{dt}{\left(t+\frac{1}{3}\right)^2 + \frac{2}{3} - \frac{1}{9}} =$$

$$= \frac{1}{3}\int \frac{dt}{\left(t+\frac{1}{3}\right)^2 + \frac{5}{9}} = \frac{1}{\cancel{3}}\cdot\frac{1\cdot\sqrt{5}/\cancel{3}}{5/\cancel{9}}\int \frac{\frac{3}{\sqrt{5}}\,dt}{\left(\frac{3\left(t+\frac{1}{3}\right)}{\sqrt{5}}\right)^2 + 1} =$$

$$= \frac{\sqrt{5}}{5}\arctan\frac{3t+1}{\sqrt{5}} \quad (+C) =$$

$$= \frac{\sqrt{5}}{5}\arctan\frac{3\tan\frac{x}{2} + 1}{\sqrt{5}} \quad (+C)$$

5.
$$\int_0^{2\pi} \sin x \sin(x+\varphi)\,dx =$$
$$= \int_0^{2\pi} (\sin^2 x \cos\varphi + \sin x \cos x \sin\varphi)\,dx =$$
$$= \cos\varphi \int_0^{2\pi} \frac{1-\cos 2x}{2}\,dx + \sin\varphi \int_0^{2\pi} \frac{\sin 2x}{2}\,dx =$$
$$= \cos\varphi \cdot \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{2\pi} +$$
$$+ \sin\varphi \cdot \frac{1}{2} \cdot \left[-\frac{\cos 2x}{2}\right]_0^{2\pi} =$$
$$= \cos\varphi \cdot \frac{1}{\cancel{2}} \cdot \cancel{2}\pi$$
$$\Rightarrow M(f) = \frac{1}{2\pi} \cdot \cos\varphi \cdot \pi = \frac{1}{2}\cos\varphi$$

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6.
$$\frac{0 < b_n}{0 < b_{n+1}} = \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} = \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n+2}{n+1}\right)^{n+1}} \cdot \frac{n+1}{n+2} =$$
$$= \frac{n+1}{n+2} \cdot \left(\frac{(n+1)^2}{n(n+2)}\right)^{n+1} = \frac{n+1}{n+2}\left(1 + \frac{1}{n(n+2)}\right)^{n+1} >$$

Bernoullis olikhet

$$\frac{n+1}{n+2}\left(1 + \frac{n+1}{n(n+2)}\right) = \frac{n(n+1)\cancel{(n+2)} + (n+1)\cancel{(n+2)}}{n(n+2)^{\cancel{2}}}$$
$$= \frac{n^2+2n+1}{n^2+2n} > 1 \Rightarrow \text{följden avtagande (strängt)}$$

$$b_n = \left(1+\frac{1}{n}\right) \cdot \left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} 1 \cdot e = e \quad \text{strängt avtagande}$$
$$\Rightarrow b_n > e$$