(a) $f(x,y) = x^3 \ln y$

$\nabla f = \langle 3x^2 \ln y, \frac{x^3}{y} \rangle$

$\nabla f(2,1) = \langle 0, 8 \rangle$

$\bar{v} = \langle 3, 1 \rangle$, $|\bar{v}| = \sqrt{3^2+1^2} = \sqrt{10}$

$D_{\bar{v}}(2,1) = \nabla f \cdot \frac{\bar{v}}{|\bar{v}|} = \langle 0, 8 \rangle \cdot \langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \rangle$

$= \frac{8}{\sqrt{10}}$

(b) The direction of $\nabla f(2,1) = \langle 0, 8 \rangle$.

$\square$

2

a) $f(x,y) = -x^2 - 2y^2 + 2x + 5$

$\nabla f = \langle -2x+2, -4y \rangle = 0$

$x = 1, \ y = 0$ $\qquad$ $(1,0)$ is a critical pt.

$f_{xx} = -2, \ f_{yy} = -4, \ f_{xy} = 0$

$D = 8 \geq 0$
$f_{xx} < 0$ $\implies$ $(1,0)$ is a local maximum.

(b) The pts of abs. max and min are either in critical pts inside D or on the boundary of D.

The critical pt $(1,0) \in D$.

$f(1,0) = \boxed{6}$.

To find pts of abs. max and min on the boundary of D, we will use Lagrange method.

$g(x,y) := x^2 + y^2 - 16$.

(2

$\nabla g = \langle 2x, 2y \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} -2x + 2 = 2\lambda x \\ -4y = 2\lambda y \\ x^2 + y^2 = 16 \end{cases} \iff \begin{cases} 1 - x = \lambda x & (1) \\ -2y = \lambda y & (2) \\ x^2 + y^2 = 16 & (3) \end{cases}$$

$(2) \Rightarrow$ either $y = 0$ or $\lambda = -1$.

From (1) $x = \frac{1}{1+\lambda}$.

Case 1 : $y = 0$.

From (3): $\frac{1}{(1+\lambda)^2} = 16$

$1 + \lambda = \pm \frac{1}{4} \Rightarrow x = \pm 4$.

Hence $(4,0)$ and $(-4,0)$ are possible pts of ~~pos~~ abs. max/min.

$f(4,0) = -16 + 8 + 5 = \boxed{-3}$

$f(-4,0) = -16 - 8 + 5 = \boxed{-19}$

Case 2 : $\lambda = -2 \Rightarrow 1 - x = -2x$

$1 = -x$

$x = -1$.

$\Rightarrow \ 1+y^2=16, \quad y=15, \quad y = \pm\sqrt{15}.$

$(-1, \sqrt{15})$ and $(-1, -\sqrt{15})$ possibly are pts of abs. max/min.

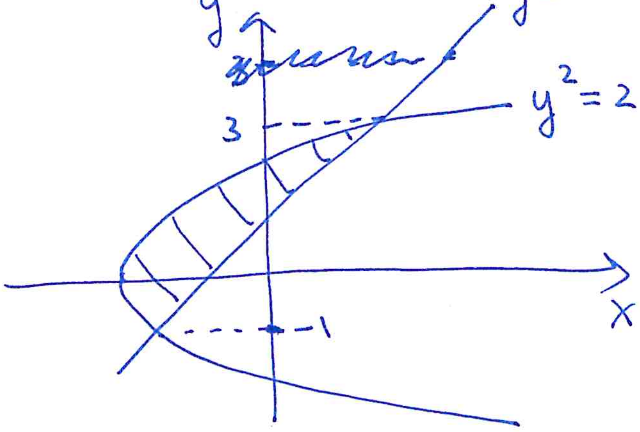
$$f(-1, \sqrt{15}) = -1-30-2+5 = \boxed{-28}$$

$$f(-1, -\sqrt{15}) = \boxed{-28}.$$

Among 5 pts of possible abs max/min we find that $(-1, \sqrt{15})$ and $(-1, -\sqrt{15})$ are pts of abs. min, and $(1, 0)$ is the pt of abs. max.

$\square$

3.



$y = x+1 \implies x = y-1$

$y^2 = 2x+5 \implies x = \frac{y^2}{2} - \frac{5}{2}$

$$y-1 = \frac{y^2}{2} - \frac{5}{2}$$

$$\frac{y^2}{2} - y - \frac{3}{2} = 0$$

$$y = -1;\ y = 3.$$

$$\iint_D y\, dA = \int_{-1}^{3} \int_{\frac{y^2}{2}-\frac{5}{2}}^{y-1} y\, dx\, dy$$

$$= \int_{-1}^{3} y\left(y - 1 - \frac{y^2}{2} - \frac{5}{2}\right) dy = \int_{-1}^{3} y\left(y - \frac{y^2}{2} + \frac{3}{2}\right) dy$$

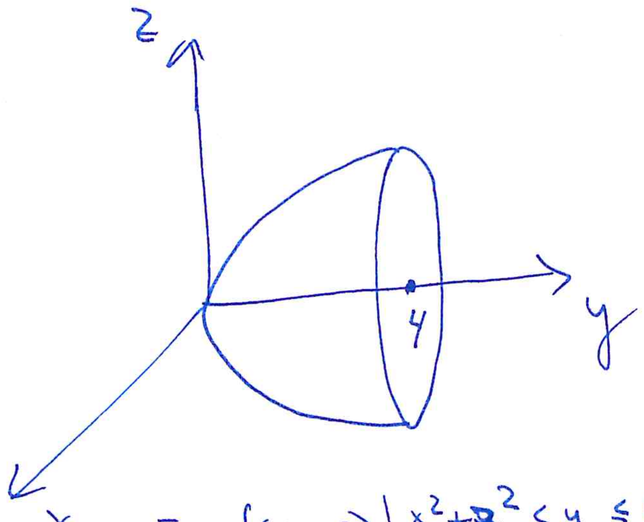
$$= \int_{-1}^{3} \left(y^2 - \frac{y^3}{2} + \frac{3}{2}y\right) dy$$

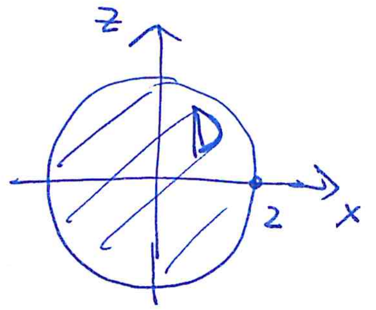
$$= \left[\frac{y^3}{3} - \frac{y^4}{8} + \frac{3y^2}{4}\right]_{y=-1}^{y=3}$$

$$= 9 - \frac{81}{8} + \frac{27}{4} + \frac{1}{3} - \frac{1}{8} - \frac{3}{4} = \frac{61}{12}.$$

$\square$.

4





$E = \{(x,y,z) \mid x^2+z^2 \le y \le 4,\ (x,z)\in D\} = \{(y,r,\theta) \mid r^2 \le y \le 4,\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$$V = \iiint_E dV \overset{\text{cylindrical coordinates}}{=\!=} \int_0^{2\pi}\int_0^2\int_{r^2}^4 r\,dy\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r(4-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (4r-r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[2r^2-\frac{r^4}{4}\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

$\square$

(5)

$$\bar{r}(t) = (4t-1)\,\bar{i} - \cos(5t)\,\bar{j} + \sin(5t)\,\bar{k},$$

$$\frac{\pi}{2} \le t \le \frac{3\pi}{2}$$

$$\bar{r}'(t) = \langle 4,\ 5\sin(5t),\ 5\cos(5t) \rangle$$

$$|\bar{r}'(t)| = \sqrt{16+25} = \sqrt{41}.$$

$$L = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} |\bar{r}'(t)|\,dt = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{41}\,dt = \pi\sqrt{41}.$$

$\square$.

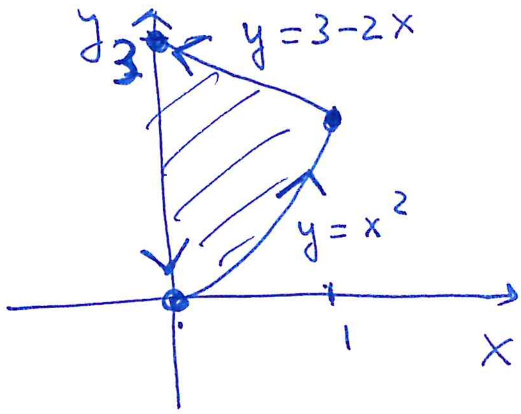
6

$$\overline{F}(x,y) = \underbrace{(x^2 y + e^x \sin(3x))}_{P}\,\overline{i} + \underbrace{x^3}_{Q}\,\overline{j}$$

(a) $\dfrac{\partial P}{\partial y} = x^2$, $\quad \dfrac{\partial Q}{\partial x} = 3x^2 \neq \dfrac{\partial P}{\partial y}$

$\Longrightarrow$ $\overline{F}$ is not conservative.

(b) $$\text{work} = \int_C \overline{F}\cdot d\overline{r} \;\overset{\text{Green's Th.}}{=\!=\!=}\; \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



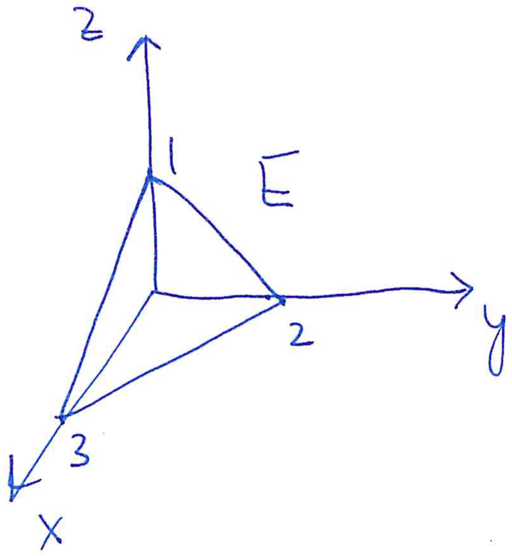
$$= \iint_D 2x^2\, dA$$

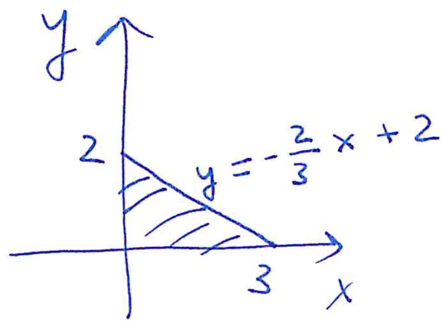
$$= \int_0^1 \int_{x^2}^{3-2x} \cancel{\frac{\partial Q}{\partial x}}\; 2x^2\, dy\, dy$$

$$= \int_0^1 2x^2(3-2x-x^2)\, dx = \int_0^1 (6x^2 - 4x^3 - 2x^4)\, dx$$

$$= \left[2x^3 - x^4 - \frac{2x^5}{5}\right]_{x=0}^{x=1} = 2 - 1 - \frac{2}{5} = \frac{3}{5}.$$

$\square$





$2x + 3y + 6z = 6.$

$z = 1 - \frac{x}{2} - \frac{y}{2}$

$$\overline{F}(x, y, z) = (y^2 z + x)\,\overline{i} + e^{z\cos x}\,\overline{j} + (y^2 - 1)\,\overline{k}$$

$\operatorname{div} \overline{F} = 1.$

$$\iint_S \overline{F} \cdot d\overline{S} \overset{\text{Gauss Th.}}{=\!=} \iiint_E \operatorname{div} \overline{F}\, dV$$

$$= \iiint_E dV = \int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{1-\frac{x}{3}-\frac{y}{2}} dz\, dy\, dx$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} \left(1 - \frac{x}{3} - \frac{y}{2}\right) dy\, dx$$

$$= \int_0^3 \left[\left(1-\frac{x}{3}\right)y - \frac{y^2}{4} \right]_{y=0}^{y=-\frac{2}{3}x+2} dx$$

$$= \int_0^3 \left(\left(1-\frac{x}{3}\right)\left(-\frac{2}{3}x+2\right) - \frac{1}{4}\left(-\frac{2}{3}x+2\right)^2 \right) dx$$

$$= \int_0^3 \left(-\frac{2}{3}x + \frac{2x^2}{9} + 2 - \frac{2x}{3} - \frac{1}{4}\left(\frac{4}{9}x^2 - \frac{8x}{3} + 4\right) \right) dx$$

$$= \int_0^3 \left(-\frac{2}{3}x + \frac{1}{9}x^2 + 1 \right) dx = \left[-\frac{x^2}{3} + \frac{x^3}{27} + x \right]_{x=0}^{x=3}$$

$$= -3 + 1 + 3 = 1.$$

□

Alternative way to compute $\iiint_E dV$ is to use that $\iiint_E dV =$ volume of E and use ~~the~~ formula for the volume of tetrahedron.

□