Lecture 1: Introduction

MVE055 / MSG810
Mathematical statistics and discrete mathematics

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GU & Chalmers University of Technology

Teachers

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Time table (1st week)

Lecture	Monday	13-16
Exercise	Tuesday	10-12
Lecture	Wednesday	10-12
Exercise	Thursday	10-12

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Course overview

https://chalmers.instructure.com/courses/15306

Examination

"För godkänd på kursen krävs godkänd på de tre grupparbetana samt godkänd på skriftlig tentamen. Betyget på kursen baseras på betyget på tentan."

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- Required for passing but does not affect course grade.

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Regression to find linear relationships between inputs/explanatory variables and outputs/explained variables.

Example: Probability vs statistics

What is the probability to throw 10 times heads in a row with a fair coin.

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What is the probability to throw 10 times heads in a row with a fair coin.

This is the 10th time you throw head in a row... is that coin fair!?

Describing data

Visual inspection

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Example:
Throwing a dice 20 times we obtained the following results:

$$1, 3, 3, 3, 1, 6, 6, 5, 1, 4, 6, 1, 4, 5, 1, 1, 2, 3, 6, 5.$$

Frequency table and histogram

Everything starts with data and tables.

If the observations take values in a small set, then we can summarise the data in a frequency table showing how many outcomes we have for each possible outcome.

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For our results

$$1, 3, 3, 3, 1, 6, 6, 5, 1, 4, 6, 1, 4, 5, 1, 1, 2, 3, 6, 5$$

we get

Outcome	1	2	3	4	5	6
Count	6	1	4	2	3	4
Proportion	0.30	0.05	0.20	0.10	0.15	0.20

Tricky denominators

ZIP	Neighborhood	Estimated Population	At Least 1 Dose	At Least 1 Dose (%)	Fully Vaccinated	Fully Vaccinated (%) ▼
10004	Financial District	2,972	3,718	100%	3,341	100%
10006	Financial District	3,382	4,087	100%	3,599	100%
10018	Hell's Kitchen/Midtown Manhattan	11,791	18,861	100%	15,089	100%
10036	Hell's Kitchen/Midtown Manhattan	27,242	35,718	100%	30,586	100%
10001, 10118	Chelsea/NoMad/West Chelsea	27,613	29,985	100%	25,988	94%
10019, 10020	Hell's Kitchen/Midtown Manhattan	43,522	45,518	100%	40,120	92%
11355	Flushing/Murray Hill/Queensboro Hill	78,853	76,759	97%	71,043	90%
10017	East Midtown/Murray Hill	15,613	15,705	100%	14,059	90%
10007	TriBeCa	6,991	6,512	93%	5,997	86%
10022	East Midtown	30,896	27,664	90%	25,509	83%

New York City Health Department, 2021-08-08.

Bar chart

Using the frequency table we can draw a bar chart. For each value we draw a bar whose height is proportional to the number of observations for that value.

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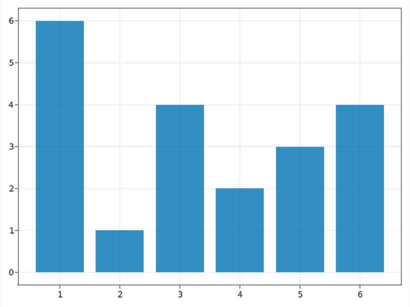


```
using StatsBase, GLMakie
x = [1, 3, 3, 3, 1, 6, ..., 5, 1, 1, 2, 3, 6, 5,]
barplot(counts(x, 1:6))
```

Histogram

Task: Summarise 1000 real numbers which are the outcome of some experiment,

$$12.15, 17.33, 0.96, 13.44, 11.27, 4.76, 8.26, 11.37, 24.31, 21.07, \ldots$$

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A bar chart doesn't make sense because the data does not have only a few different values. We can use a histogram:

- Divide the data into a number of classes (intervals) and then calculate the number of observations in each class.

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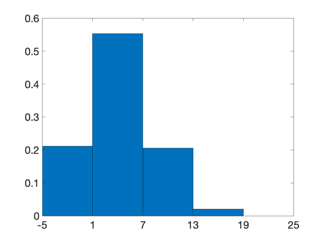
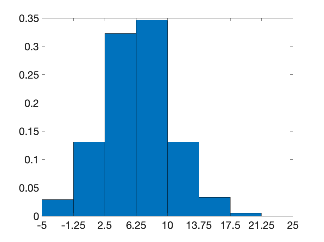
- Divide the data into a number of classes (intervals) and then calculate the number of observations in each class.
- Draw bars where the height is proportional to the number of observations in the class and the width equals the interval width.

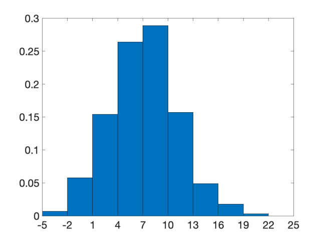
Histogram



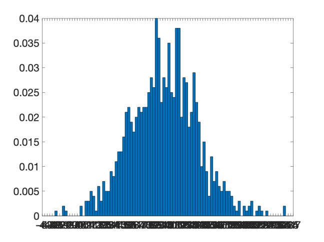
4 classes



7 classes



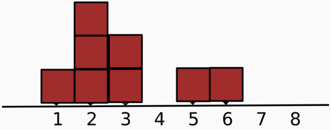
9 classes



200 classes

Sample statistics for location

Case	1	2	3	4	5	6	7	8
Value	2	3	2	6	5	1	2	3



Weights on a bar

Sample median

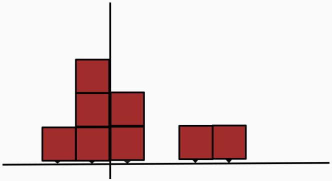
- To obtain the sample median,
  write the values in sorted order and take the middle one.

  If there is an even number of values in the data set, take the average of the two middle most.

Median

Median

Value	1	2	2	2	3	3	5	6



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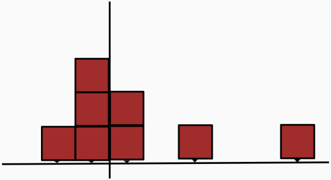

Median = 2.5

Sample mean

- The (sample) mean, denoted as $\bar{x}$, can be calculated as

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

where $x_1, x_2, \cdots, x_n$ are the n observed values.

In words: Sum the values of all cases in the data set and divide by the total number of values.

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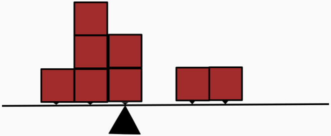

$$\text{Mean } \bar{x} = \frac{1\cdot 1 + 3\cdot 2 + 2\cdot 3 + 1\cdot 5 + 1\cdot 6}{8} = 3$$

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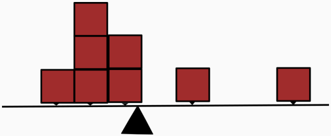

$$\text{Mean } \bar{x} = \frac{1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 5 + 1 \cdot 8}{8} = 3.25$$

Sample statistics for variation/spread

Sample variance: The sample variance of a data set $x_1, \ldots, x_n$ is given by

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}((x_1 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2)$$

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Sample standard deviation s: the square root $\sqrt{s^2}$ of the sample variance.

Example 1 (cont.)

For the dice throw example

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we obtain the mean

$$\bar{x} = (1 + 3 + 3 + \ldots + 3 + 6 + 5)/20 = 67/20 = 3.35$$

Sorting the values and taking the central one we obtain the median 3.

The variance is

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$$s^2 = ((1 - 3.35)^2 + (3 - 3.35)^2 + \ldots + (5 - 3.35)^2)/19 = 3.8184$$

and the standard deviation is $s = 1.9541$.

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2. Ask a person on the street which party they would vote for.
3. Throw a handful of coins and count the heads.
4. Examine a unit from a manufacturing process.
5. Measure the round-trip time (ping) of a connection.

The result of the experiment is called outcome ω (*utfall*). The set of possible outcomes is called the sample space Ω (*utfallsrummet*).
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Sample spaces

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- $\Omega = [0, \infty)$ (seconds).

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We group outcomes into events.

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An event A occurs if any of the outcomes $\omega \in A$ occurs in the experiment.

Example

Outcome and sample space

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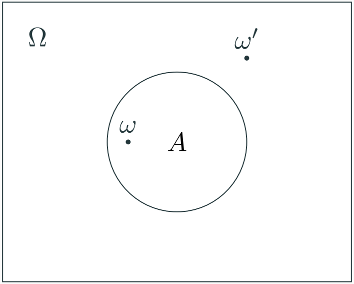
We like events because the probability of a single outcome might be too small or zero.

Event, outcome and sample space



Event A, outcome $\omega \in A$ and sample space Ω

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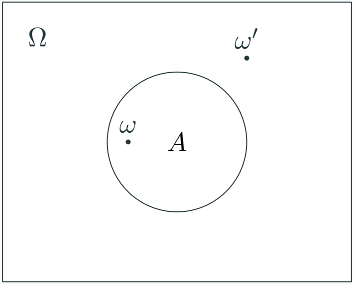

Event A, outcome $\omega \in A$ and sample space Ω

And some other outcome $\omega' \notin A$.

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Overview: Intersection, union and complement

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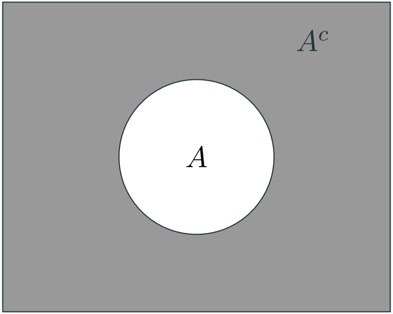
Mutually exclusive events

If $A \cap B = \varnothing$ then A and B are mutually exclusive events.

Example: The set $\{2, 4, 6\}$ and the set $\{1, 3, 5\}$ are disjoint.

Complement

Complement



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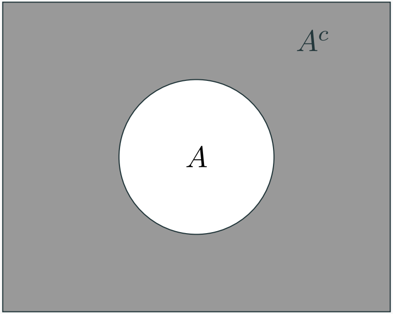

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$$A^c = \Omega \backslash A.$$

In the example with the die: Here $A = \{1, 3, 5\}$. So if the die shows a 2, then $A^c = \{2, 4, 6\}$ happened.

Union

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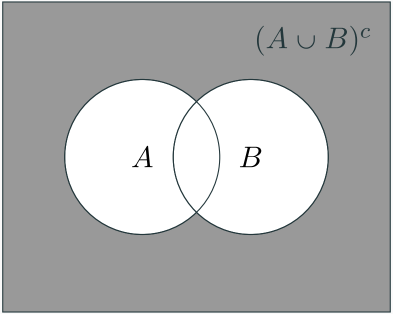
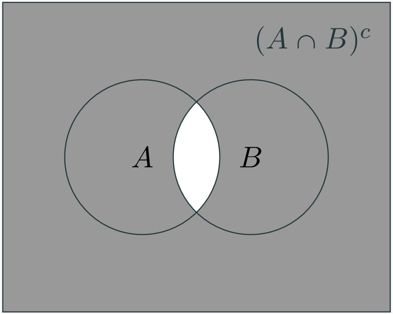


If we have events, A and B we can define $A \cup B$, the union of A and B .

- $A \cup B$ occurs if A or B occur (or both).

Intersection

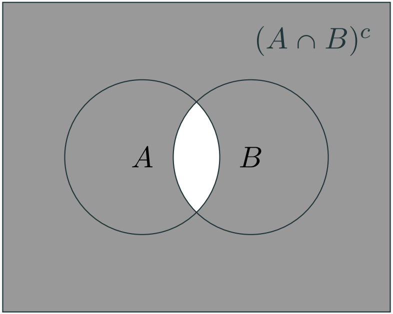
Intersection



The intersection $A \cap B$ are all elements both in A and B.

- So for $A \cap B$ to occur, both A and B need to occur.

Intersection

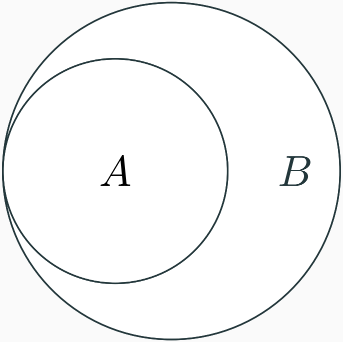


The intersection $A \cap B$ are all elements both in A and B.

- So for $A \cap B$ to occur, both A and B need to occur.

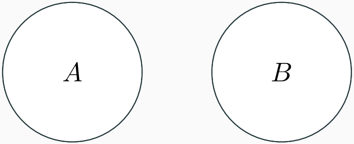
$A \cap B = \varnothing$ means that A and$B$ exclude each other.

Set inclusion



$A \subset B$.

Disjoint sets



$$A \cap B = \varnothing.$$

The empty set $\varnothing$

Permutations and combinations

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A specific order of a number of objects.

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Combination

A selection of objects without regard for their order.

$\{1, 3, 5\}$ is a combination of 3 the of the numbers 1 to 6.

Note $(1, 2) \neq (2, 1)$ **but** $\{1, 2\} = \{2, 1\}$.

Permutations and combinations

Multiplication principle

If there are a ways to make a choice and there are b ways to make a second choice, then there are ab ways to make a combined choice.

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For $n \in \mathbb{N}$ define $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ and $0! = 1$. $n!$ is read "n-factorial".

$$4! = \boxed{}$$

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Calculate the number of combinations

Number of combinations

The number of ways we can choose r objects out of a total of n distinct objects, ignoring their order, is given by

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

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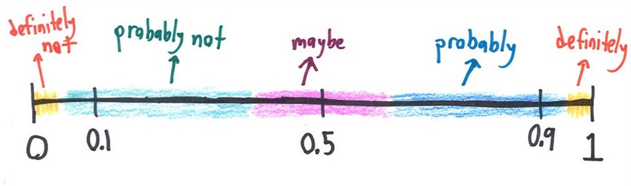
Example: Draw five cards from a poker set of 52 cards. $2\,598\,960$ combinations are possible:

$$ \binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2\,598\,960 $$

${}_nP_r = \frac{n!}{(n-r)!}$

Probabilities of events

- Probability is a numerical measure of how likely an event is to happen.



- Probability is a *proportion*, a number between 0 and 1. Notation

$$\mathrm{P}(\textit{something that can happen}) = \textit{a probability}.$$

E.g.

$$\mathrm{P}(\text{coin heads-up}) = \frac{1}{2}.$$

Figure from https://mathwithbaddrawings.com/.

Equally likely outcomes

What is probability?

Equally likely outcomes

What is probability? (How do we assign probability?)

- A classical and useful view considers equally likely outcomes. Then

$$\mathsf{P}(A) = \frac{\text{number of outcomes for which } A \text{ occurs}}{\text{total number of outcomes}}$$

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- A classical and useful view considers equally likely outcomes. Then

$$\mathsf{P}(A) = \frac{\text{number of outcomes for which } A \text{ occurs}}{\text{total number of outcomes}}$$

- Probability to throw an odd number with a fair die.

$$\mathsf{P}(A) = \frac{|\{1,3,5\}|}{|\{1,2,3,4,5,6\}|} = \frac{3}{6} = \frac{1}{2}$$

Frequentist interpretation of probability

- Sometimes it is not reasonable to assume that all outcomes are equally likely.

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- The frequentist interpretation of probability: Suppose we repeat a random experiment many times under identical conditions. As the number of repetitions n grows, we observe that the proportion n_A/n of times that an event A occur converges to a number. This number is the probability of A, or as formula

$$\frac{n_A}{n} \to \mathsf{P}(A), \text{where } n \to \infty$$

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$$\frac{n_A}{n} \rightarrow \mathsf{P}(A), \text{where } n \rightarrow \infty$$

Example: With a fair die, we observe the proportion of times where $A = \{\text{even number of eyes}\}$ occurs converge to $\frac{1}{2}$.

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Kolmogorov's axioms

A probability measure P is function $A \mapsto \mathsf{P}(A)$ assigning each event $A \subset \Omega$ a probability,

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A probability measure P is function $A \mapsto \mathsf{P}(A)$ assigning each event $A \subset \Omega$ a probability,a positive number such that

1. $0 \leqslant \mathsf{P}(A) \leqslant 1$.
2. $\mathsf{P}(\Omega) = 1$.
3. For pairwise disjoint events $A_1, A_2, \ldots$

$$\mathsf{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathsf{P}(A_i).$$

Especially for disjoint/mutually exclusive events A and B,

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B).$$

Properties of probability distributions

The axioms determine all further properties of probabilities...

Properties

For the probability measure P it holds that:

1. $\mathsf{P}(\varnothing) = 0$.
2. $\mathsf{P}(A^c) = 1 - \mathsf{P}(A)$.
3. $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$.

All these properties can be seen with the help of Venn diagrams.

Probability of the union of non-disjoint events

What is the probability of drawing a jack or a red card from a well shuffled full deck (52 cards)?

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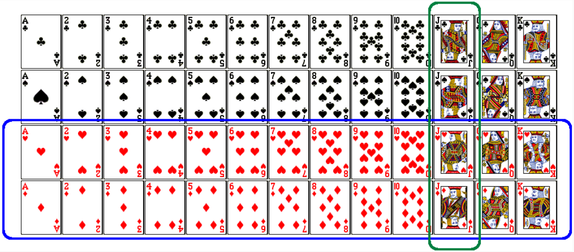

$$\mathrm{P}(\text{jack or red}) = \mathrm{P}(\text{jack}) + \mathrm{P}(\text{red}) - \mathrm{P}(\text{jack and red})$$
$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

Figure from `http://www.milefoot.com/math/discrete/counting/cardfreq.htm`.

General addition rule

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

Combined experiment

Throw a coin (1, ), and throw a 6 sided die. What is

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Throw a coin (①,), and throw a 6 sided die. What is

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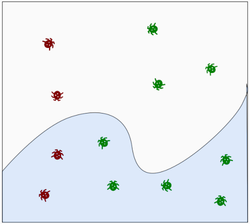
Use multiplication rule and the classical approach.

	⚀	⚁	⚂	⚃	⚄	⚅	
①	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

The table also shows the *marginal probablities.*

Example with the bugs

Drawing a random bug out of the aquarium, with (g)reen and (r)ed bugs on (l)and and (w)ater.



	R	G	
L	2	3	5
W	2	5	7
	4	8	12

	R	G	
L	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{5}{12}$
W	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{7}{12}$
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Frequency table and probability table.

Flawed reasoning

Students at an elementary school are given a questionnaire that they are required to return after their parents have completed it.

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One of the questions asked is, "Do you find that your work schedule makes it difficult for you to spend time with your kids after school" Of the parents who replied, 85% said "no".

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Students at an elementary school are given a questionnaire that they are required to return after their parents have completed it.

One of the questions asked is, “Do you find that your work schedule makes it difficult for you to spend time with your kids after school” Of the parents who replied, 85% said “no”.

Based on these results, the school officials conclude that a great majority of the parents have no difficulty spending time with their kids after school.

What went wrong?

Conditional probability

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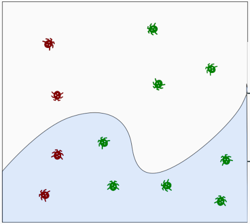

	R	G	
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$$\mathrm{P}(\text{is red}) = 1/3$$

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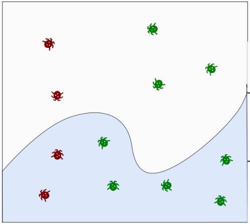

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We catch a red bug. What is the probability it is "dry": $\boxed{}$

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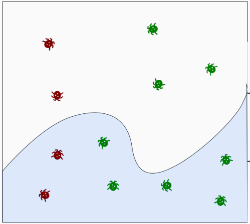

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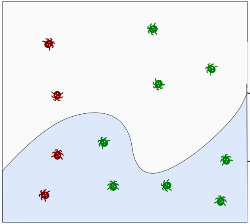

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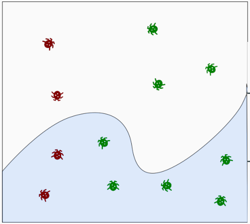

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$$\mathrm{P}(\text{lives on land} \mid \text{is red}) = \frac{\mathrm{P}(\text{red and land}}{\mathrm{P}(\text{is red})} = \frac{2/12}{4/12}$$

Conditional probability

The *conditional probability* of the event of interest A given condition B is calculated as

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

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Multiplication rule

If A and B represent two events, then

$$P(A \cap B) = P(A|B) \cdot P(B)$$

Note that this formula is simply the conditional probability formula, rearranged.