Matematisk Statistik och Disktret Matematik, MVE055/MSG810, HT19

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Lärare

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Kurslitteratur

(MA) J. Milton, J. Arnold, Introduction to Probability and Statistics 4th ed McGraw-Hill

(GS) C. Grinstead, J Snell, Introduction to Probability AMS (on-line).

(EG) E. Eriksson, H. Gavel, Diskret matematik, Studentlitteratur, ISBN 9144028784. Relevanta delar av boken finns på Canvas.

(A) J. Anderson, J. Bell, J. Anderson, Discrete Mathematics with Combinatorics. Vi använder bara några övningar; dessa kan hittas på Canvas.

Examination

Examinationen har två moment

- Skriftlig tentamen i slutet av kursen
- Inlämningsuppgifter

Inlämningsuppgifter:

- Tre obligatoriska inlämningsuppgifter.
- Grupper på 1-3 personer.
- Uppgifterna lämnas in via Canvas.
- Deadline: 22 september, 6 oktober och 20 oktober.

1. Introduction to Probability and Counting

2. Probability laws

1. Introduction to Probability and Counting

What is probability?

What is the probability that an open heart surgery will be successful?
What is the probability to win the lottery?
What is the probability that it will be sunny tomorrow?

The probability is a quantitative measure of the likelihood of an event to happen.

Definitions

- **Outcome** *(sv. utfall)*: Result of a random trial.
- **Sample space** *(sv. utfallsrum)*: The set S of all possible outcomes.
- **Event** ***(sv. händelse)***: A collection of outcomes, a subset of S.
- The empty set is called the **impossible event** and is denoted by $\varnothing$.
- S is called the **certain event**.
- Two events A och B are said to be **disjoint** *(sv. disjunkta)* if $A \cap B = \varnothing$.
- The events $A_1, A_2, \ldots$ are said to be **mutually disjoint** *(sv. parvis disjunkta)* if $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Example

If we roll a fair six-sided die, we have six possible outcomes 1,2,3,4,5,6. The sample space is therefore $S = \{1, 2, 3, 4, 5, 6\}$*.*
A={The die shows an odd number } och B={The die shows a number less than 3} are events. A and B are written as follows: $A = \{1, 3, 5\}$ *and* $B = \{1, 2\}$

Example

If we roll two dice at the same time, then

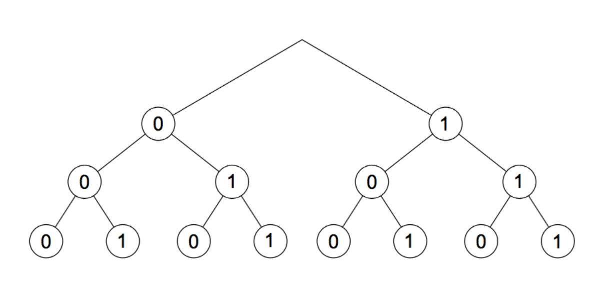
$$S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$$

C="The sum of the numbers is at most 3" = $\{(1, 1), (1, 2), (2, 1)\}$

The outcome of a trial are sometimes represented by a **tree.**

Example

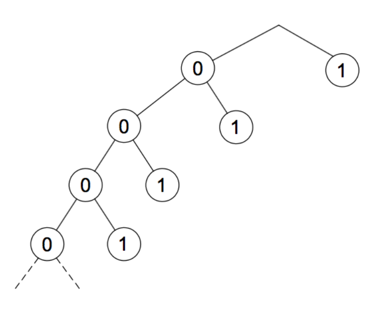
A coin is flipped 3 times. We denote Tail by 0 and Head by 1.



$S = \{000, 001, 010, 011, 100, 101, 110, 111\}$

Example

A coin is flipped until we get Head for the first time.



$S = \{1, 01, 001, 0001, \dots\}$.

Combinatorics - Multiplication principle

Multiplication principle: Assume that an event takes place in k consecutive steps. Suppose that the i-th step occurs in n_i different ways. Then, the number of possible ways for the event to occur is $\Pi_{i=1}^{k} n_i = n_1 \cdot n_2 \cdots n_k$.

Example

9 men and 7 women were invited to a party. If the men dance only with women and women dance only with me, we have $9 \cdot 7 = 63$ *different couples.*

Combinatorics - Permutation

How many possible numbers can one build from $\{1, 2, 3\}$ without repeating the same digit twice?
According to the multiplication principle there are $3 \cdot 2 \cdot 1 = 6$ possible outcomes:

$$123, 132, 213, 231, 312, 321.$$

Definition

A **permutation** is the act of arranging the elements of a set into a sequence.

To get the number of permutations of n elements we proceed as follows:
The first element has n choices, the second element has $n-1$ choices since the element chosen first cannot be chosen again, and so on. The total number of possibilities is therefore:

$$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = \Pi_{i=1}^{n} = n!.$$

$n!$ is called n-fatorial. $0! = 1$ by convention.

Theorem

The number of permutations of n elements is equal to $n!$.

Combinatorics - Arrangements *(sv. ordnad urval)*

How many two-digits number can one get from the set $\{1,2,3,4,5\}$ without repeating the same digit twice?
By the multiplication principle there exists $5 \cdot 4$ possibilities which is equal to $\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = \frac{5!}{(5-2)!}$.

Theorem (Arrangements)

The number of ways to permute r elements chosen from a set of n elements is denoted by ${}_nP_r$ and is given by

$$ {}_nP_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!} $$

Combinatorics - Combination

- The number of ways to pick r elements from a set of n elements where the order of the elements is not important is equal to $\frac{{}_nP_r}{r!}$ and is denoted by ${}_nC_r$
- ${}_nC_r = \frac{n!}{(n-r)!r!}$
- ${}_nC_r$ is called the binomial coefficient and is also denoted by $\binom{n}{r}$.

Example

There exist $\binom{52}{2} = \frac{52!}{2!50!} = \frac{52 \cdot 51}{2}$ *different ways to pick two cards from a deck of cards.*

Combinatorics - Permutation with repetition

How many words can we get if we permute the letters in the word “DADDY”?

The number of permutations of 5 letters is 5!. For each of these permutations, we can permute the 3 letters D in 3! ways but the arrangement is unchanged. Therefore, in order to count the different possible permutations, we divide 5! by 3!.

In general, if S has k distinct elements where the first element is repeated n_1 times, the second is repeated n_2 times, $\ldots$, the k-th element is repeated n_k times, and $n_1 + \ldots n_k = n$, then the number of permutations of the elements in S is given by

$$\binom{n!}{n1, n2, \cdots n_k} = \frac{n}{n_1!n_2!\cdots n_k!}.$$

Probability

A probability is a number between 0 and 1 that describes how likely an event to occur. If the event is denoted by A, the probability that A occurs is denoted by $p(A)$ or $P(A)$.

- The probability of the impossible event is 0 ($p(\varnothing) = 0$). Probabilities near zero indicate that the event is not very likely to occur.
- The probability of the certain event is 1 ($p(S = 1)$. Probabilities near 1 are very likely to occur.

Relative frequency *(sv. Frekventistiska Approximation)*

Suppose an experiment was run n times. The probability that an event A occur is approximated by

$$p(A) = \frac{n_A}{n} = \frac{\text{number of times A occur}}{\text{number of times the experiment was run}}$$

This probability is based on experience and the approximation is not accurate when n is too small.

Example

The following tabel gives the outcome of 10 rolls of a die. We are interested of the event A="The die shows the number 6".

Försök	Resultat (antal ögon)	Händelse A	Relativ frekvens
1	5	Nej	0/1
2	6	Ja	1/2
3	2	Nej	1/3
4	3	Nej	1/4
5	4	Nej	1/5
6	4	Nej	1/6
7	1	Nej	1/7
8	6	Ja	2/8
9	5	Nej	2/9
10	1	Nej	2/10

If we repeat the experiment many times we see that the relative frequency tends to 1/6.

Classical probability *(sv. Klassiska sannolikhet)*

Suppose now that an experiment has n possible outcomes that are equally likely to occur and n_A is the number of possible outcomes of the event A. The classical probability theory says that the probability that the event A occur i

$$p(A) = \frac{n_A}{n}$$

Example

In the previous example, $A = \{6\}$ *and* $S = \{1, 2, 3, 4, 5, 6\}$*. Therefore,*

$$p(A) = \frac{1}{6}.$$

Probability laws

Let S be the sample space and A and B be two events.

- **Complement**: The complement event to A is the event A'="A does not occur." (denoted by A' or $\bar{A}$ or A^c)

$$p(A') = 1 - p(A).$$

- **Addition rule**

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

- If A and B are disjoint then, $\Rightarrow p(A \cap B) = 0$ and $p(A \cup B) = p(A) + p(B)$

Example

Let A and B be two events such that $p(A) = 0.5$, $P(B) = 0.7$ and $p(A \cap B) = 0.4$. Find $p(A \cup B)$, $p(A \cap B')$, $p(A' \cap B)$ and $p(A' \cap B')$.

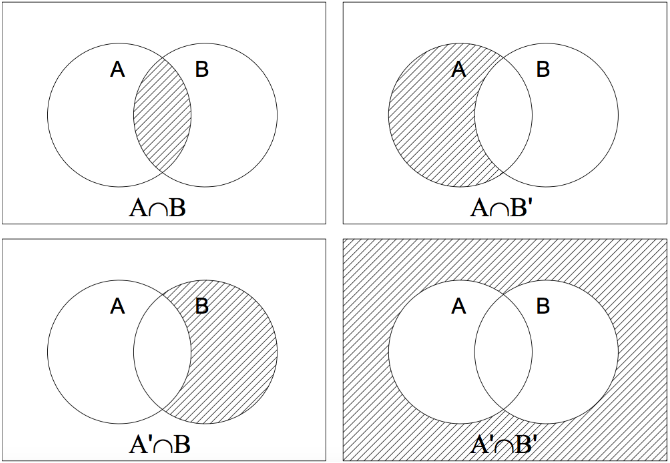
Solution

$p(A \cup B) = p(A) + p(B) - p(A \cap B) = 0.5 + 0.7 - 0.4 = 0.8$

$p(A \cap B') = p(A) - p(A \cap B) = 0.5 - 0.4 = 0.1$

$p(A' \cap B) = p(B) - p(A \cap B) = 0.7 - 0.4 = 0.3$

$p(A' \cap B') = p((A \cup B)') = 1 - p(A \cup B) = 1 - 0.8 = 0.2$


A
B
A∩B
A
B
A∩B'
A
B
A'∩B
A
B
A'∩B'

Conditional probability *(sv. Betingad sannolikhet)*

Example

300 products in a factory have been chosen randomly for a quality control and they were classified either as "defected" or "good". Some of these products were produced by an old machine and the others by a new one. The following table gives the results of the experiment.

	Good	Defected	Total
Old machine	170	10	180
New machine	115	5	120
Total	285	15	300

Example

A product is chosen randomly. Let A, B, and C be three events defined as follows A="The chosen product is good".
B="The chosen product is produced by an old machine.
C="The chosen product is good knowing that (given that) is was produced by an old machine."
$p(A) = \frac{285}{300}$, $p(B) = \frac{180}{300}$
The event C invovles both events A and B, and is written as $C = A|B$ *(A given B, or A knowing B). From the table we can get the probability of C by considering only the row of the old machine, namely,*

$$p(C) = p(A|B) = \frac{170}{180} = \frac{p(A \cap B)}{p(B)}$$

Conditional probability

Given two events A and B with $p(B) \neq 0$. The **conditional probability** that A occurs given that B has occured is defined by

$$p(A|B) = \frac{p(A \cap B)}{p(B)}.$$

Multiplication rule

Multiplication rule: Suppose that $p(A) \neq 0$ and $p(B) \neq 0$.

$$p(A \cap B) = p(A|B)p(B) = p(B|A)p(A).$$

The conditional probability can therefore be written as

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

Independent events *(sv. Oberoende händelser)*

Independent events: Assume that the information "B has occured" has no influence on the probability that A occur, then $p(A|B) = p(A)$, and the multiplication rule can be written as $p(A \cap B) = p(A)p(B)$.

Theorem

A and B are said to be independent if and only if

$$p(A \cap B) = p(A)p(B).$$

This is equivalent to say that $p(A|B) = p(A)$ if $p(B) \neq 0$ and $p(B|A) = p(B)$ if $p(A) \neq 0$.

Example

At the entrance to a casino, there are two slot machines. Machine A is programmed so that in the long run it will produce a winner in 10% of the plays. Machine B is programmed so that in the long run it will produce a winner in 15% of the plays. The two machines run independently of each other. If we play each machine once, what is the probability that we will win on at least one play?

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) = p(A) + p(B) - p(A)p(B) = 0.235$$

Bayes' theorem

Example

The fire alarm in a certain company seems to be reliable. However, a false alarm can sometimes occur or a fire could be missed. Let F="A fire has occured" och A="The alarm starts beeping.", $p(F) = 0.05$, $p(A|F) = 0.98$, and $p(A|F') = 0.10$. Find $p(F|A)$.

Lösning:

$$p(F|A) = \frac{p(F \cap A)}{p(A)} = \frac{p(A|F)p(F)}{p(A)}$$

Example

To compute $p(A)$ *we can use the formula*

$$A = (A \cap F) \cup (A \cap F')$$

Since $A \cap F$ *and* $A \cap F'$ *are disjoint, then*

$$\begin{aligned} p(A) &= p(A \cap F) + p(A \cap F') \\ &= p(A|F)p(F) + p(A|F')p(F') \end{aligned}$$

Therefore

$$\begin{aligned} p(F|A) &= \frac{p(A|F)p(F)}{p(A|F)p(F) + p(A|F')p(F')} \\ &= \frac{0.98 \cdot 0.05}{0.98 \cdot 0.05 + 0.10(1 - 0.05)} \\ &= 0.34 \end{aligned}$$

Bayes' theorem

Let $A_1, A_2, \cdots A_n$ be mutually disjoint events such that their union is S and $B \neq \emptyset$ be an event. For all A_j, $j = 1, \cdots n$

$$p(A_j|B) = \frac{p(B|A_j)p(A_j)}{\sum_{i=1}^{n} p(B|Ai)p(A_i)}$$