

# Lecture 7: Markov chains

MVE055 / MSG810

Mathematical statistics and discrete mathematics

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# Markov chains

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The weather in the land Oz is R (rainy), S (sunny) or C (cloudy).  
Weather of the last 30 days:

*RRRRRRRCCSCCSRCSRRRRRCRSCSCCRCRS...*

What do you expect for the weather of tomorrow?

# Markov chains

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	<i>R</i>	<i>S</i>	<i>C</i>
<i>R</i>	4/7	1/7	2/7
<i>S</i>	2/5	0	3/5
<i>C</i>	3/10	4/10	3/10

## Markov chains

There, they never have two nice days in a row and if it was C (cloudy) yesterday, there is a 0.25 probability of R (rain) today.

For each day, the weather of the next day is random and we represent the probabilities by a matrix

	$R$	$S$	$C$
$R$	0.5	0.25	0.25
$S$	0.5	0	0.5
$C$	0.25	0.25	0.5

Each *row* contains the probability for next days weather depending on current weather.

## Markov chain

A Markov chain consists of:

A set of states:  $\{s_1, \dots, s_n\}$ .

A matrix of transition probabilities

$$\mathbf{P} = \begin{pmatrix} p_{11} & \dots & p_{1n} \\ \vdots & & \\ p_{n1} & \dots & p_{nn} \end{pmatrix}$$

containing the probability  $p_{ij}$  to move from state  $s_i$  to state  $s_j$

— — — —

sv: övergångssannolikhet, övergångsmatrisen

## Markov property

The transition probability does only depend on the current state:

$$p_{ij} = P(\text{next state is } s_j \mid \text{current state is } s_i \text{ and the state before ....})$$

# Transition probabilities

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Transition probabilities are conditional probabilities:

$$p_{ij} = \text{P}(\text{next state is } s_j \mid \text{current state is } s_i)$$

That means **rows** sum to 1:  $\sum_{\text{all } j} p_{ij} = 1.$



# What is the weather in three days

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The probability that the Markov chain, starting in states  $s_i$  , will be in state  $s_j$  after  $n$  steps is given by the  $ij$ 'th entry of

$$\mathbf{P}^n = \mathbf{P} \cdot \dots \cdot \mathbf{P}$$

( $n$ -fold matrix product.)

## Example

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Suppose we want to compute the probability that, given that it is rainy today, the weather will be cloudy in two days.

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$$\begin{aligned} p_{13}^{(2)} &= p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33} \\ &= 0.5(0.25) + 0.25(0.5) + 0.25(0.5) = 0.375 \end{aligned}$$

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$$\mathbf{P}^2 = \begin{pmatrix} 0.4375 & 0.1875 & \mathbf{0.375} \\ 0.375 & 0.25 & 0.375 \\ 0.375 & 0.1875 & 0.4375 \end{pmatrix}$$

# Probability vectors

A **probability vector** is a row vector that gives the probabilities of being at each state at a certain step.

The probability vector which represents the initial state of a Markov chain is starting vector and is denoted by  $\mathbf{u}^{(0)}$  or simply  $\mathbf{u}$ . The probability vector at step  $k$  is denoted by  $\mathbf{u}^{(k)}$ .

## 1 step

If  $\mathbf{u}_k$  is the probability vector at step  $k$ , then the vector

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} \mathbf{P}$$

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## n steps

If  $\mathbf{u}$  is the starting vector of a Markov Chain, then the probability vector at step  $n$  is given by

$$\mathbf{u}^{(n)} = \mathbf{u} \mathbf{P}^n.$$

## Example

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In the previous example, if the initial probability vector is  $\mathbf{u} = (1/3, 2/3, 0)$ , then the probability vector on day 2 will be

$$\mathbf{u}^{(2)} = \mathbf{u}\mathbf{P}^2 = \begin{pmatrix} 1/3 & 2/3 & 0 \end{pmatrix} \begin{pmatrix} 0.4375 & 0.1875 & 0.375 \\ 0.375 & 0.25 & 0.375 \\ 0.375 & 0.1875 & 0.4375 \end{pmatrix}$$

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This means that on day 2, there is a 39.58% chance of rain, 22.92% chance that the weather will be nice and 37.5% chance that it will be cloudy.

A Markov chain is said to be regular if there exists  $n$  such that all the elements of the matrix  $\mathbf{P}^n$  are nonzero. The Markov chain of the previous example is regular since

$$\mathbf{P}^2 = \begin{pmatrix} 0.4375 & 0.1875 & 0.375 \\ 0.375 & 0.25 & 0.375 \\ 0.375 & 0.1875 & 0.4375 \end{pmatrix}$$

(all the values are strictly positive)

## Stationary distribution

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If the Markov chain is regular then,  $\mathbf{P}^n \rightarrow \mathbf{Q}$  where

$$\mathbf{Q} = \begin{pmatrix} q_1 & q_2 & \dots & q_n \\ q_1 & q_2 & \dots & q_n \\ \vdots & \vdots & \ddots & \vdots \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$$

$q_j$  is the probability to be at state  $s_j$  on the long run.

$$q\mathbf{P} = q$$

## Absorbing states

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A state is said to be absorbing if it is impossible to leave it, that is  $p_{ii} = 1$ .

A Markov chain is called absorbing if it has at least one absorbing state, and if from every state it is possible to go to an absorbing state.

In an absorbing Markov chain, a state that is not absorbing is called transient.

Example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}$$

The transition matrix of an absorbing Markov chain with  $r$  absorbing states and  $t$  transient states can be written as

$$P = \begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ 0 & \mathbf{I}_r \end{pmatrix}$$

where  $\mathbf{I}_r$  is the identity matrix, 0 is the zero matrix (all elements are zeros),  $\mathbf{Q}$  is a  $t \times t$  -matrix and  $\mathbf{R}$  is a  $t \times r$  nonzero matrix.

This form is called the canonical form.  $\mathbf{P}^n = \begin{pmatrix} \mathbf{Q}^n & \star \\ 0 & \mathbf{I}_r \end{pmatrix}$  where  $\star$  is a  $t \times r$  matrix.  $\mathbf{Q}^n$  gives the probability for being in each of the transient states after  $n$  steps for each possible transient starting state.