Generating functions.

Def: Let $\{a_k\}_0^\infty$ be a series. The generating function corresponding to $\{a_k\}_0^\infty$ is defined by

$$g(x) = \sum_{k=0}^{\infty} a_k x^k$$

Ex. • Let $a_k = 1$ for all $k \geq 0$. Then the generating function of a_k is

$$g(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1$$

• Let $a_k = 2^k$ for $k \geq 0$. The generating function for a_k is

$$g(x) = \sum_{k=0}^{\infty} 2^k x^k = \sum_{k=0}^{\infty} (2x)^k = \frac{1}{1-2x}, \quad |2x| < 1$$

Common generating functions

1. Let $a_k = c^k$, $k \geq 0$. Then $g(x) = \sum_{k=0}^{\infty} c^k x^k = \dfrac{1}{1-cx}$, $|cx| < 1$.

2. Let $a_k = \binom{n}{k}$, $k \geq 0$ and n fixed. Then,

$$g(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$$

Recall that $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

3. Let $a_n = \binom{n+k}{k}$, $n \geq 0$ and k fixed. Then,

$$g(x) = \sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}$$

Generating functions can be used to find the number of integer solutions to equations of type $y_1 + y_2 + \cdots + y_k = n$.

Ex. Suppose we have n apples that should be distributed to 3 people, Alice, John and Aya, such that Alice wants at most 2 apples, each of John and Aya wants at least one, John cannot eat more than 3 and Aya wants at most 2. In how many ways can we divide these apples.

The problem is equivalent to solve

$$y_1 + y_2 + y_3 = n \ , \quad \begin{cases} 0 \le y_1 \le 2 \\ 1 \le y_2 \le 3 \\ 1 \le y_3 \le 2 \end{cases}$$

where y_1, y_2 and y_3 are the number of apples that Alice, John and Aya can respectively have.

The answer is the coefficient of x^n in the following generating function:

$$f(x) = \underbrace{(x^0 + x^1 + x^2)}\,(x^1 + x^2 + x^3)\,(x^1 + x^2)$$

the exponents are the possible values of y_1

$\{1,2,3\}$ are the possible values of y_2

$\{1,2\}$ are the possible values of y_3.

$$f(x) = x^2 + 3x^3 + 5x^4 + 5x^5 + 3x^6 + x^7.$$

If $n=3$, there are 3 ways of dividing the apples, namely

$(0,1,2)$ $(0,2,1)$ and $(1,1,1)$

If $n=4$, there are 5 ways of dividing the apples:

$(0,2,2)$ $(0,3,1)$ $(1,1,2)$ $(1,2,1)$ $(2,1,1)$

Ex. How many integer solutions has the equation:

$$y_1 + y_2 + y_3 = 15 \quad \text{where} \quad y_1 \leq 5 \quad \text{and} \quad y_2 \geq 5.$$

The number of solutions is the coefficient of x^{15} in the function:

$$f(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(x^5 + x^6 + x^7 + \dots)(1 + x + x^2 + \dots)$$

$$= \frac{1-x^6}{1-x} \cdot \frac{x^5}{1-x} \cdot \frac{1}{1-x} = (x^5 - x^{11}) \cdot \frac{1}{(1-x)^3}$$

Using generating functions, $\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} \binom{n+2}{2} x^n$

$$\Rightarrow f(x) = (x^5 - x^{11}) \sum_{n=0}^{\infty} \binom{n+2}{2} x^n = \sum_{n=0}^{\infty} \binom{n+2}{2} x^{n+5} - \sum_{n=0}^{\infty} \binom{n+2}{2} x^{n+11}$$

Hence, the coefficient of x^{15} is: $\binom{12}{2} - \binom{6}{2} = 51.$

(take $n = 10$ in $\binom{n+2}{2} x^{n+5}$ and $n = 4$ in $\binom{n+2}{2} x^{n+11}$)

Solve a recursive equation using generating functions.

Suppose we have the following recursive series.

$$\begin{cases} a_1 = 0 \\ a_{n+1} = 8a_n + 9 \cdot 10^{n-1} & n \geq 1 \end{cases}$$

and we wish to find an explicit formula for a_n using generating functions:

$$a_{n+1} = 8a_n + 9 \cdot 10^{n-1} \qquad n \geq 1$$

$$\Rightarrow a_{n+1} x^{n+1} = 8 a_n x^{n+1} + 9 \cdot 10^{n-1} x^{n+1} \qquad \text{(multiply both sides by } x^{n+1}\text{)}$$

$$\Rightarrow \sum_{n \geq 1} a_{n+1} x^{n+1} = \sum_{n \geq 1} \left(8 a_n x^{n+1} + 9 \cdot 10^{n-1} x^{n+1}\right)$$

$$\Rightarrow \sum_{n\geq 2} a_n x^n = 8x \sum_{n\geq 1} a_n x^n + 9x^2 \sum_{n\geq 1} 10^{n-1} x^{n-1}$$

$$\Rightarrow \sum_{n\geq 1} a_n x^n - \underset{\overset{\shortparallel}{0}}{a_1} x = 8x \sum_{n\geq 1} a_n x^n + 9x^2 \sum_{n\geq 1} (10x)^{n-1}$$

Let $S = \sum_{n\geq 1} a_n x^n$

$$\Rightarrow S = 8xS + 9x^2 \cdot \frac{1}{1-10x}$$

$$\Rightarrow S - 8xS = \frac{9x^2}{1-10x} \Rightarrow (1-8x)S = \frac{9x^2}{1-10x}$$

$$\Rightarrow S = \frac{9x^2}{(1-10x)(1-8x)} = 9x^2\left(\frac{A}{1-10x} + \frac{B}{1-8x}\right)$$

$$\frac{1}{(1-10x)(1-8x)} = \frac{A}{1-10x} + \frac{B}{1-8x} \Rightarrow \left.\frac{1}{1-8x}\right|_{x=\frac{1}{10}} = A \Rightarrow A = 5$$

and $\left.\frac{1}{1-10x}\right|_{x=\frac{1}{8}} = B \Rightarrow B = -4$

$$\Rightarrow S = 9x^2\left(\frac{5}{1-10x} - \frac{4}{1-8x}\right)$$

$$= 9x^2 \cdot \left(5\sum_{n\geq 0}(10x)^n - 4\sum_{n\geq 0}(8x)^n\right)$$

$$\Rightarrow \sum_{n\geq 1} a_n x^n = \sum_{n\geq 0}\left(45 \cdot 10^n x^{n+2} - 36 \cdot 8^n x^{n+2}\right) = \sum_{n\geq 2}\left(45 \cdot 10^{n-2} x^n - 36 \cdot 8^{n-2} x^n\right)$$

since $a_1 = 0$

$$\Rightarrow \sum_{n\geq 1} a_n x^n = \sum_{n\geq 2} a_n x^n = \sum_{n\geq 2}\left(45 \cdot 10^{n-2} - 36 \cdot 8^{n-2}\right) x^n$$

$$\Rightarrow a_n = 45 \cdot 10^{n-2} - 36 \cdot 8^{n-2}$$