Solutions (3.32 - 7.38 - 7.39 - 7.40 - 7.46 - 11.10 - 11.20)

(MA).

3.32 $m_X(t) = e^{2(e^t - 1)}$

(a) $m'_X(t) = e^{2(e^t-1)} \cdot 2e^t$

$m'_X(0) = e^{2(e^0-1)} \cdot 2e^0 = 2 \Rightarrow E[X] = 2.$

(b) $E[X^2] = m''_X(0).$

$m''_X(t) = \left(e^{2(e^t-1)} \cdot 2e^t\right) \cdot 2e^t + 2e^t e^{2(e^t-1)}$

$m''_X(0) = e^{2(e^0-1)} \cdot 2e^0 \cdot 2e^0 + 2e^0 e^{2(e^0-1)}$

$= 4 + 2 = 6$

(c) $\sigma^2 = E[X^2] - E[X]^2 = 6 - 4 = 2.$

$\sigma = \sqrt{2}.$

7.38 $Y = \alpha + \beta X$

$m_Y(t) = E[e^{tY}] = E[e^{(\alpha+\beta X)t}]$

$= E[e^{\alpha t} \cdot e^{\beta X t}] = e^{\alpha t} E[e^{X(\beta t)}]$ since $e^{\alpha t}$ is a constant with respect to X.

$m_X(t) = E[e^{Xt}] \Rightarrow m_X(\beta t) = E[e^{X(\beta t)}]$

Hence, $m_Y(t) = e^{\alpha t} m_X(\beta t).$

7.39 $\quad Y = a_0 + a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$

$$m_Y(t) = E[e^{Yt}] = E\left[e^{(a_0 + a_1 X_1 + \cdots + a_n X_n)t}\right]$$

$$= E\left[e^{a_0 t} \cdot e^{a_1 X_1 t} \cdot \ldots\ldots e^{a_n X_n t}\right]$$

By independency,

$$= E[e^{a_0 t}] \cdot E[e^{a_1 X_1 t}] \cdot \ldots\ldots E[e^{a_n X_n t}]$$

$$= e^{a_0 t}\, m_{X_1}(a_1 t) \cdot m_{X_2}(a_2 t) \ldots m_{X_n}(a_n t)$$

$$= e^{a_0 t} \prod_{i=1}^{n} m_{X_i}(a_i t).$$

7.40 $\quad Y = 3X_1 + 6X_2 - 8$

$$m_Y(t) = e^{-8t}\, m_{X_1}(3t)\, m_{X_2}(6t)$$

(Use 7.39 with $n=2$, $a_0=-8$, $a_1=3$ and $a_2=6$)

$$X_1 \sim N(2, 9) \Rightarrow m_{X_1}(t) = e^{2t + 9t^2/2}$$
$$\Rightarrow m_{X_1}(3t) = e^{6t + 81t^2/2}$$

$$X_2 \sim N(5, 1) \Rightarrow m_{X_2}(t) = e^{5t + t^2/2}$$
$$\Rightarrow m_{X_2}(6t) = e^{30t + 36t^2/2}$$

Hence, $m_Y(t) = e^{-8t}\, e^{(6t + 81t^2/2)}\, e^{(30t + 36t^2/2)}$

$$= e^{-8t + 6t + 30t + 81t^2/2 + 36t^2/2}$$

$$= e^{28t + 117t^2/2} \qquad \Rightarrow Y \sim N(28, 117)$$

7.46 $X \sim N(10, 3^2)$ $Y \sim N(9, 4^2)$

(a) $X - Y$ is normally distributed

$E[X-Y] = E[X] - E[Y] = 10 - 9 = 1$

$V[X-Y] = V[X+(-Y)] = V[X] + V[-Y]$

$= V[X] + (-1)^2 V[Y]$

$= V[X] + V[Y] = 3^2 + 4^2 = 25$

$\Rightarrow X - Y \sim N(1, 25)$.

(b) $P(X < Y) = P(X - Y < 0) = P\left(\frac{X-Y-1}{5} < \frac{-1}{5}\right)$

$= P\left(\frac{X-Y-1}{5} < -0{,}2\right) = 0{,}4207$

11.10 (b) $\sum_{i=1}^{24} x_i = 89{,}25$ $\sum_{i=1}^{24} x_i^2 = 520{,}6875$ $\sum_{i=1}^{24} x_i y_i = 2{,}728 \times 10^5$

$\sum_{i=1}^{24} y_i = 46\,720$ $\sum_{i=1}^{24} y_i^2 = 14\,292\,9462$

$$b_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \sum x_i^2} = 524{,}72.$$

$$b_0 = \bar{y} - b_1 \bar{x} = \frac{\sum y_i}{24} - b_1 \frac{\sum x_i}{24} = -4{,}6231$$

$\Rightarrow$ Linear regression line: $\mu_{Y|x} = -4{,}6231 + 524{,}72\, x$

(c) $x = 3{,}45 \Rightarrow y = 524{,}72 \cdot 3{,}45 - 4{,}6231 = 1805{,}6$

11.20

$$b_0 = \bar{y} - b_1 \bar{x} \quad , \quad b_1 = \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2}$$

$$\sum x_i = 464{,}4 \quad \Rightarrow \quad \bar{x} = \frac{464{,}4}{8} = 58{,}05.$$

$$\sum y_i = 46{,}2 \quad \Rightarrow \quad \bar{y} = \frac{46{,}2}{8} = 5{,}775$$

$$\sum x_i^2 = 32090 \qquad \sum x_i y_i = 3173{,}2$$

$$\Rightarrow b_1 = 0{,}0957 \quad , \quad b_0 = 0{,}2177$$

A C.I. on B_0 is given by

$$b_0 \pm t_{\frac{\alpha}{2}} \, s \sqrt{\frac{\sum x_i^2}{n S_{xx}}}$$

$$S_{xx} = \left(n \sum x_i^2 - (\sum x_i)^2\right)/n = 5131{,}5$$

$$t_{\frac{\alpha}{2}} = 2{,}306 \qquad (df = 6)$$

$$S^2 = \frac{SSE}{n-2} = \frac{S_{yy} - b_1 S_{xy}}{n-2} = 8{,}0892$$

$$\Rightarrow \text{C.I.:} \left(0{,}2177 \pm 2{,}306 \cdot \sqrt{\frac{8{,}0892\,(32090)}{8\,(5131{,}5)}}\right)$$

$$= (-5{,}581 \; ; \; 6{,}0163)$$