Lectures

MVE055 / MSG810
Mathematical statistics and discrete mathematics

Moritz Schauer

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GU & Chalmers University of Technology

Teachers

Moritz Schauer	Instructor
Room:	H3029
E-mail:	smoritz@chalmers.se
Erik Jansson	Teaching assistant
E-mail:	erikjans@chalmers.se
Samuel Winqvist	Teaching assistants
E-mail:	samwin@chalmers.se

Time table (1st week)

Lecture	Monday	HB1	15-17
Exercise	Tuesday	ML14, ML15,	10-12
Lecture	Wednesday	HB3	10-12
Exercise	Thursday	ML14, ML15	10-12

Student representatives

oskararnudd@gmail.com	Oskar Arnudd
axel.carlstedt99@gmail.com	Axel Carlstedt
tim.le.karlsson@gmail.com	Tim Karlsson
cajsa.t@hotmail.se	Cajsa Tellström
alice.thornell@yahoo.se	Alice Thornell

Course overview

`https://chalmers.instructure.com/courses/20092`

Examination

“För godkänd på kursen krävs godkänd på de tre grupparbetana samt godkänd på skriftlig tentamen. Betyget på kursen baseras på betyget på tentan.”

Examination consists of two parts.

Exam:

- Exam takes place on campus. **Will** look similar to the last exam.

3 group assignments:

- Due 2022-09-19, 2022-10-10, 2022-11-07.
- First assignment: “Skiplist”.
- Groups of up to four students.
- ↳ Find yourself a group on canvas "Project groups".
- One student hands in for the group on canvas.
- Required for passing but does not affect course grade.

Course content

In **probability theory** we construct and analyse mathematical models for phenomena that exhibit uncertainty and variation. Highlight: Markov chains.

In **statistics** we observe data and we want to infer the probabilistic model or parameters of such a model: **inverse probability.**

Generating functions allow to solve recursive equations.

The law of large number describes what happens if you perform the same experiment a large number of times.

Regression to find linear relationships between inputs/explanatory variables and outputs/explained variables.

Example: Probability vs statistics

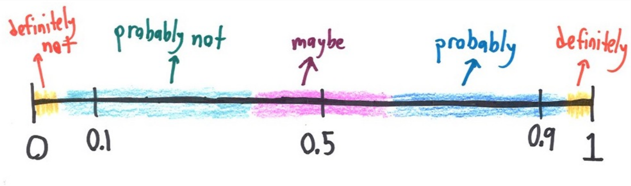
What is the probability to throw 10 times heads in a row with a fair coin.

This is the 10th time you throw head in a row... is that coin fair!?

Probabilities

Probabilities of events

- Probability is a numerical measure of how likely an event is to happen.



- Probability is a *proportion*, a number between 0 and 1. Notation

$$\mathrm{P}(\textit{something that can happen}) = \textit{a probability}.$$

E.g.

$$\mathrm{P}(\text{coin lands heads-up}) = \frac{1}{2}.$$

Figure from `https://mathwithbaddrawings.com/`.

Equally likely outcomes

What is probability? (How do we assign probability?)

- A classical and useful view considers equally likely outcomes. Then

$$\mathsf{P}(A) = \frac{\text{number of outcomes for which event } A \text{ occurs}}{\text{total number of outcomes}}$$

- Probability to throw an odd number with a fair die.

$$\mathsf{P}(A) = \frac{|\{1,3,5\}|}{|\{1,2,3,4,5,6\}|} = \frac{3}{6} = \frac{1}{2}$$

All outcomes: "1,2,3,4,5,6". Event: $A = \{1,3,5\}$ ("odd number") occurs for outcomes $1,3,5$.

Frequentist interpretation of probability

- Sometimes it is not reasonable to assume that all outcomes are equally likely.
- The frequentist interpretation of probability: Suppose we repeat a random experiment many times under identical conditions. As the number of repetitions n grows, we observe that the proportion n_A/n of times that an event A occurs converges to a number.

  This number is the probability of A, or as formula

$$\frac{n_A}{n} \to \mathrm{P}(A), \text{where } n \to \infty$$

Example: With a fair six-sided die, we observe the proportion of times where $A = \{2, 4, 6\}$ occurs converge to $\frac{1}{2}$.

Sample spaces

Outcomes

In probability theory we consider experiments which have non-deterministic, variable or random outcomes. For example

1. Roll a die and count the eyes.
2. Throw a handful of coins and count the heads.
3. Examine a unit from a manufacturing process.
4. Measure the round-trip time (ping) of a connection.

The result of the experiment is called outcome ω (*utfall*). The set of possible outcomes is called the sample space Ω (*utfallsrummet*).
$\hookrightarrow$ Sets Ω and their elements ω.

Sample spaces

- $\Omega = \{$1,2,3,4,5,6$\}$.
- $\Omega = \{$(head, head),(head, tail),(tail, head),(tail, tail)$\}$ (for 2 coins).
- $\Omega = \{$defect, intact$\}$.
- $\Omega = [0, \infty)$ (seconds).

Events

We group outcomes into events.

An event A is a set of outcomes, that is, a subset of the sample space Ω.

Example for events:

1. $A = \{1,3,5\}$, that is "my die shows an odd number".
2. $A = \{(\text{head,head}),(\text{tail,tail})\}$, "both coins show the same face".
3. $A = \{\text{defect}\}$, the "unit is broken".
4. $A = \{x : x \geqslant 0.5\}$, round-trip-time larger than 0.5s.

An event A **occurs** if any of the outcomes $\omega \in A$ occurs in the experiment.

↳ Sets Ω, their elements ω and subsets A, B, ...

Outcome and sample space

Event, outcome and sample space

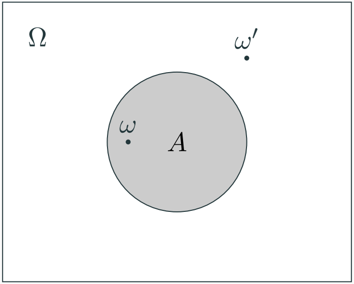
The *outcome* ω is the result of a random experiment, and the set of all possible outcomes Ω is called the *sample space*.

Events

An event is a collection (a set of) different outcomes. The event A, as a set of outcomes, is therefore a subset of the sample space Ω.

We like events because the probability of a single outcome might be too small or zero.

Event, outcome and sample space



Event A, outcome $\omega \in A$ and sample space Ω

And some other outcome $\omega' \notin A$.

Overview: Intersection, union and complement

For events A and B we have defined:

Complement, A^c

Set of all outcomes ω not contained in A. $A^c = \Omega \backslash A$.

Union, $A \cup B$

Set of all outcomes ω in A or B.

Intersection, $A \cap B$

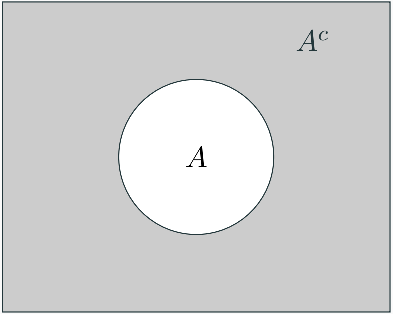
Set of all outcomes ω in A and B.

A^c, $A \cup B$, $A \cap B$ are also events. $\varnothing$ and Ω are also events, the **impossible** event and the **sure** event.

Mutually exclusive events

If $A \cap B = \varnothing$ then A and B are **mutually exclusive** events.

Complement

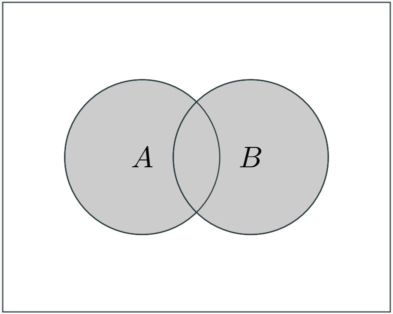


The complement of a A are all outcomes not in A.

$$A^c = \Omega \backslash A.$$

In the example with the die: Here $A = \{1, 3, 5\}$. So if the die shows a 2, then $A^c = \{2, 4, 6\}$ happened.

Union

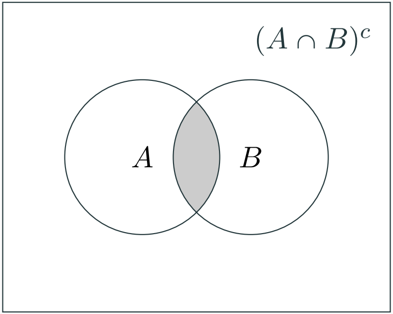


If we have events, A and B we can define $A \cup B$, the union of A and B .

- $A \cup B$ occurs if A or B occur (or both).

Example: $\{2, 4, 6\} \cup \{1, 2, 3\} = \{1, 2, 3, 4, 6\}$ are disjoint.

Intersection

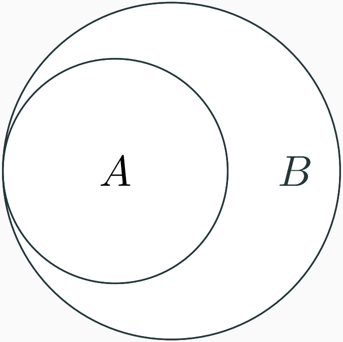


The intersection $A \cap B$ are all elements both in A and B.

- So for $A \cap B$ to occur, both A and B need to occur.

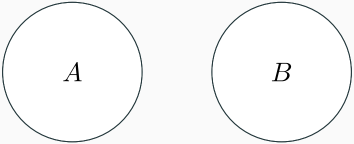
$A \cap B = \varnothing$ means that A and$B$ exclude each other.

Set inclusion



$A \subset B$.

Disjoint sets



$$A \cap B = \varnothing.$$

Example: The set $\{2, 4, 6\}$ and the set $\{1, 3, 5\}$ are disjoint.

The empty set $\varnothing$

Permutations and combinations

Permutation

A specific order of a number of objects.

$(1, 3, 2, 5, 6, 4)$ is a permutation of the numbers 1 to 6.

Combination

A selection of objects without regard for their order.

$\{1, 3, 5\}$ is a combination of 3 the of the numbers 1 to 6.

Note $(1, 2) \neq (2, 1)$ **but** $\{1, 2\} = \{2, 1\}$.

Permutations and combinations

Multiplication principle

If there are a ways to make a choice and there are b ways to make a second choice, then there are ab ways to make a combined choice.

If you draw cards from a deck of 52, then you can choose between 52 cards for your first draw, between the remaining 51 cards for your next draw, etc.

Factorial

For $n \in \mathbb{N}$ define $n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$ and $0! = 1$. $n!$ is read "n-factorial".

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Calculate the number of r-permutations

Number of r-permutations

The number of ways we can choose r objects in order out of a total of n distinct objects is given by

$$_nP_r = \frac{n!}{(n-r)!}$$

Example: Draw a ordered sequence of 5 cards from a poker set of 52 cards.

$$_{52}P_5 = \frac{52!}{47!} = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311\,875\,200$$

Calculate the number of combinations

Number of combinations

The number of ways we can choose r objects out of a total of n distinct objects, ignoring their order, is given by

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- $_nC_r$ is usually called binomial coefficient.

Example: Draw five cards from a poker set of 52 cards. $2\,598\,960$ combinations are possible:

$$\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2\,598\,960$$

Compare $_nP_r = \frac{n!}{(n-r)!}$