Lectures

MVE055 / MSG810
Mathematical statistics and discrete mathematics

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Last updated August 31, 2022

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Kolmogorov's axioms

Let Ω be a sample space.

Kolmogorov's axioms

A probability measure P is function $A \mapsto \mathsf{P}(A)$ assigning each event $A \subset \Omega$ a probability, a positive number such that

1. $0 \leqslant \mathsf{P}(A) \leqslant 1$.
2. $\mathsf{P}(\Omega) = 1$.
3. (Simplified) For disjoint/mutually exclusive events A and B

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B)$$

Recall: $A \cup B = A$ or B, $A \cap B = A$ and B.

Properties of probability distributions

The axioms determine all further properties of probabilities...

Properties

For the probability measure P it holds that:

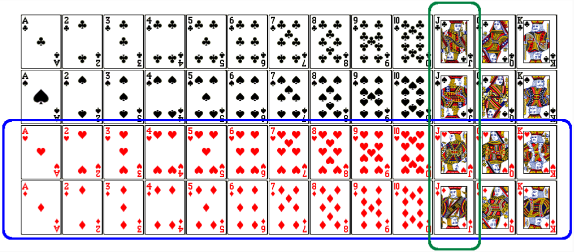
1. $\mathsf{P}(\varnothing) = 0$.
2. $\mathsf{P}(A^c) = 1 - \mathsf{P}(A)$.
3. $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$.

All these properties can be seen with the help of Venn diagrams.

Recall: $A \cup B = A$ or B, $A \cap B = A$ and B.

Probability of the union of non-disjoint events

What is the probability of drawing a jack or a red card from a well shuffled full deck (52 cards)?



$$\mathrm{P}(\text{jack or red}) = \mathrm{P}(\text{jack}) + \mathrm{P}(\text{red}) - \mathrm{P}(\text{jack and red})$$
$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52}$$

Figure from `http://www.milefoot.com/math/discrete/counting/cardfreq.htm`.

General addition rule

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

Independent events

Independent events

Two events A and B are independent if $\mathsf{P}(A \cap B) = \mathsf{P}(A)\mathsf{P}(B)$.

Simple example: Throw a 6-sided die. Are $A = \{5, 6\}$ and $B = \{1, 3, 5\}$ independent?

$$\mathrm{P}(A)\mathrm{P}(B) = \frac{2}{6} \cdot \frac{3}{6} = \frac{1}{6}, \quad \mathrm{P}(A \cap B) = \mathrm{P}(\{5\}) = \frac{1}{6}.$$

Yes!

Combined experiment

Throw a coin with sides , ① and throw a 6 sided die. What is the probability of the event "throw and a ⚃ together"?

Form a sample space of all pairs and use the classical approach.

$$\Omega = \{(\text{}, ⚀), (\text{}, ⚁), (\text{}, ⚂), (\text{}, ⚃), (\text{}, ⚄), (\text{}, ⚅),$$
$$(①, ⚀), (①, ⚁), (①, ⚂), (①, ⚃), (①, ⚄), (①, ⚅)\}.$$

So our event is...

$$A = \{(\text{}, ⚃)\}$$

$$\mathrm{P}(A) = \frac{|A|}{|\Omega|} = \frac{1}{12}$$

Combined experiment

Create a table:

	⚀	⚁	⚂	⚃	⚄	⚅	
	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
①	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{2}$
	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	1

The table also shows the **marginal probabilities** e.g.

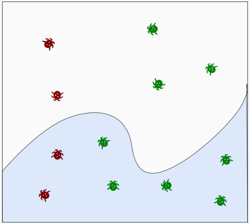
$$\mathrm{P}(\text{4 eyes}) = \mathrm{P}(\{(\;, ⚃), (①, ⚃)\}) = \frac{2}{12} = \frac{1}{6}.$$

Note

$$\mathrm{P}(\text{4 eyes and heads}) = \frac{1}{12}, \quad \mathrm{P}(\text{4 eyes})\mathrm{P}(\text{heads}) = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}.$$

Example with the bugs

Drawing a random bug out of the aquarium, with (g)reen and (r)ed bugs on (l)and and (w)ater.



	R	G	
L	2	3	5
W	2	5	7
	4	8	12

	R	G	
L	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{5}{12}$
W	$\frac{1}{6}$	$\frac{5}{12}$	$\frac{7}{12}$
	$\frac{1}{3}$	$\frac{2}{3}$	1

Frequency table and probability table.

Flawed reasoning

Students at an elementary school are given a questionnaire that they are required to return after their parents have completed it.

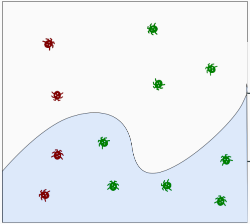
One of the questions asked is, "Do you find that your work schedule makes it difficult for you to spend time with your kids after school" Of the parents who replied, 85% said "no".

Based on these results, the school officials conclude that a great majority of the parents have no difficulty spending time with their kids after school.

What went wrong?

Conditional probability

Drawing a random bug out of the aquarium, with (g)reen and (r)ed bugs on (l)and and (w)ater.

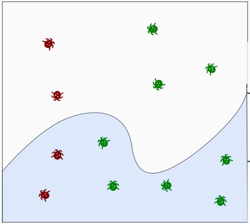


	R	G	
L	$1/6$	$1/4$	$5/12$
W	$1/6$	$5/12$	$7/12$
	$4/12$	$8/12$	1

$$\mathrm{P}(\text{is red}) = 1/3$$

Conditional probability

Drawing a random bug out of the aquarium, with (g)reen and (r)ed bugs on (l)and and (w)ater.



	R	G	
L	$1/6$	$1/4$	$5/12$
W	$1/6$	$5/12$	$7/12$
	$4/12$	$8/12$	1

We catch a red bug. What is the probability it is "dry": 50%-50%

$$\mathrm{P}(\text{lives on land} \mid \text{is red}) = \frac{\mathrm{P}(\text{red and land}}{\mathrm{P}(\text{is red})} = \frac{2/12}{4/12}$$

Conditional probability

The *conditional probability* of the event of interest A given condition B is calculated as

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

Multiplication rule

If A and B represent two events, then

$$P(A \cap B) = P(A|B) \cdot P(B)$$

Note that this formula is simply the conditional probability formula, rearranged.

Conditional distribution

If we know some event B occurs, the probability of A given the new information B can be calculated as follows:

Conditional probability

Assume that $\mathsf{P}(B) > 0$. The conditional probability of A given B is defined as

$$\mathsf{P}(A \mid B) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)}. \tag{0.1}$$

Multiplication rule for probabilities

For events A and B it holds

$$\mathsf{P}(A \cap B) = \mathsf{P}(B \mid A)\mathsf{P}(A) = \mathsf{P}(A \mid B)\mathsf{P}(B).$$

The multiplication rule is useful to calculate probabilities of multiple events affecting each other.

Bayes formula

For events A and B

$$\mathsf{P}(A \mid B) = \frac{\mathsf{P}(B \mid A)\mathsf{P}(A)}{\mathsf{P}(B)}$$

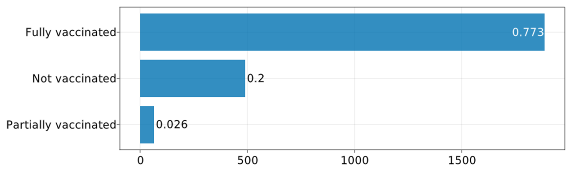
Often it is useful to rewrite the denominator $\mathsf{P}(B)$

$$\mathsf{P}(B) = \mathsf{P}(B \mid A)\mathsf{P}(A) + \mathsf{P}(B \mid A^c)\mathsf{P}(A^c)$$

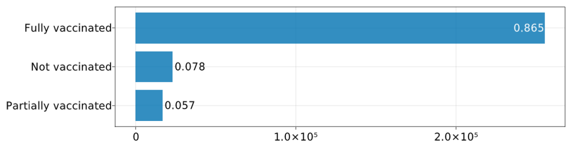
Base rate fallacy

Base rate fallacy

In the fourth wave (July 10 - August 16, 2021) about 2400 (or 0.825 %) people 16 or older in Island have been diagnosed with Covid-19:



But (young) adults in Iceland's population are highly vaccinated



$$\mathrm{P}(\text{diagn} \mid \text{vacc}) = \frac{\mathrm{P}(\text{vacc} \mid \text{diagn})\mathrm{P}(\text{diagn})}{\mathrm{P}(\text{vacc})} = \frac{0.773 \cdot 0.00825}{0.864} = 0.00738$$

Base rate fallacy

$$\mathrm{P}(\mathsf{diagn} \mid \mathsf{vacc}) = \frac{0.773 \cdot 0.00825}{0.864} = 0.00738$$
$$\mathrm{P}(\mathsf{diagn} \mid \mathsf{not\ vacc}) = \frac{0.200 \cdot 0.00825}{0.0783} = 0.0211$$

$$\mathrm{P}(\mathsf{diagn} \mid \mathsf{part.\,vacc}) = \frac{0.0262 \cdot 0.00825}{0.0570} = 0.00379\ (\mathsf{sic!})$$

Independent events

Two events A and B are independent if knowing whether B occured does not change the probability of A

$$\mathsf{P}(A \mid B) = \mathsf{P}(A).$$