Lectures

MVE055 / MSG810
Mathematical statistics and discrete mathematics

Moritz Schauer

Last updated September 5, 2022

GU & Chalmers University of Technology

Sample mean

Sample mean

- The (sample) mean, denoted as $\bar{x}$, can be calculated as

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

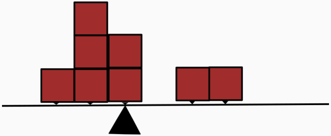
where $x_1, x_2, \cdots, x_n$ are the n observed values.

In words: Sum the values of all cases in the data set and divide by the total number of values.

Sample mean

Mean

Value	1	2	2	2	3	3	5	6

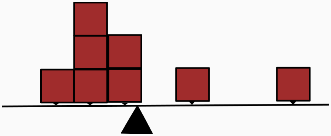


Mean $\bar{x} = \dfrac{1\cdot 1 + 3\cdot 2 + 2\cdot 3 + 1\cdot 5 + 1\cdot 6}{8} = 3$

Sample mean

Mean

Value	1	2	2	2	3	3	5	8



Mean $\bar{x} = \dfrac{1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 1 \cdot 5 + 1 \cdot 8}{8} = 3.25$

Random variables

Random variables

Random variables

A random variable is a numeric quantity whose value depends on the outcome of a random experiment.

Example: X is the number of eyes on a 6-sided die.

We denote random variables with capital letters, often X or Y.

Examples?

Pair of dice

Throw a pair of dice, count the total number of eyes, call that random variable X. Consider the **event** that $X = 7$.

Event? What are the actual ω making our event and sample space Ω? You could take

$$\Omega = \{⚀⚀, \dots, ⚅⚅\}$$

Then the set of $\omega \in \Omega$ where $X = 7$ is

$$A = \{⚀⚅, ⚁⚄, ⚂⚃, ⚃⚂, ⚄⚁, ⚅⚀\}$$

Therefore

$$\mathsf{P}(X = 7) = \mathsf{P}(A) = \frac{|A|}{|\Omega|} = \frac{6}{36}$$

Value k	2	3	4	5	6	7	8	9	10	11	12
Probability $\mathsf{P}(X{=}k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Pair of dice

The following holds

$$\mathrm{P}(X = x) = \begin{cases} \frac{6-|k-7|}{36} & \text{if } x \in \{2, \dots, 12\} \\ 0 & \text{otherwise.} \end{cases}$$

Check:

Value x	2	3	4	5	6	7	8	9	10	11	12	other
Probability $\mathrm{P}(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0

Discrete random variables

Discrete random variables

A random variable is called discrete if it is integer-valued or otherwise has only a finite or countable number of values.

Example: $Y = X/2$ is discrete (but can take non-integers such as $Y = 5.5$ as values.)

Probability mass function

Probability mass function

Define the probability mass function f of a discrete random variable X by

$$f(x) = \mathsf{P}(X = x).$$

So $f(x) = 0$ for all real x such that $\mathsf{P}(X = x) = 0$, okay?

Sometimes we write f_X to talk about X's own probability mass function.

Sum of two dice

$$f(x) = \begin{cases} \frac{6-|x-7|}{36} & \text{if } x \in \{2, 3, \ldots, 12\} \\ 0 & \text{otherwise} \end{cases}$$

is the probability mass function for the random variable which counts the sum of two dice.

Two coins

Flip two coins... count the number of heads. Call it X.

$f(0) = \frac{1}{4}, f(1) = \frac{1}{2}$ and $f(2) = \frac{1}{4}$

$f(x) = 0$ otherwise if $x \notin \{0, 1, 2\}$.

Flip two coins... count the number of heads. $f_X(0) = \frac{1}{4}$, $f_X(1) = \frac{1}{2}$ and $f_X(2) = \frac{1}{4}$.

What is $\mathsf{P}(X \in \{1, 2\}) = \mathsf{P}(1 \leqslant X \leqslant 2)$?

$$\mathsf{P}(1 \leqslant X \leqslant 2) = f_X(1) + f_X(2) = \frac{3}{4}$$

Let $Y = X/2$. What is $\mathsf{P}(Y > 0)$?

$$\mathsf{P}(Y > 0) = \mathsf{P}(1 \leqslant X \leqslant 2) = f_X(1) + f_X(2) = \frac{3}{4}$$

Rule

For integer valued X

$$\mathsf{P}(m \leqslant X \leqslant n) = \sum_{k=m}^{n} f(k)$$

for any integers m and n.

Describing distributions

Probability mass function

Not all functions are probability mass functions. Because they describe probability distributions, some conditions must hold.

$f(x)$ is a probability mass function if and only if

- $f(x) \geqslant 0$ for all x.
- $\sum_{\text{all } x} f(x) = 1$.

If somebody gives you a probability mass function, there is a random variable for it.

Distribution function

Assume X is a discrete random variable. Its distribution function is given by

$$F_X(x) = \mathsf{P}(X \leqslant x) = \sum_{\text{all } k \leqslant x} f_X(k),$$

Flip two coins... count the number of heads. Call it X.
$f(0) = \frac{1}{4}, f(1) = \frac{1}{2}$ and $f(2) = \frac{1}{4}$. Find F.

$$\begin{aligned}
F(0) &= f(0) = \tfrac{1}{4} \\
F(1) &= f(0) + f(1) = \tfrac{1}{4} + \tfrac{1}{2} \\
F(2) &= f(0) + f(1) + f(2) = 1
\end{aligned}$$

Distribution function

What is the probability to throw k times heads in a row with a fair coin?

$$f(0) = \frac{1}{2}, \quad f(1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}, \quad f(2) = \frac{1}{8}, \quad f(k) = \left(\frac{1}{2}\right)^{k+1}$$

$$\mathsf{P}(X > 0) = f(1) + f(2) + f(3) + ... = 1 - \mathsf{P}(X = 0) = 1 - f(0)$$

Distribution function

For $F(x)$ it holds

- $F(x)$ is increasing
- $F(x) \to 1$ for $x \to \infty$.
- $F(x) \to 0$ for $x \to -\infty$.

Distribution function

Also

- $\mathsf{P}(a < X \leqslant b) = F(b) - F(a)$.
- $\mathsf{P}(X > a) = 1 - F(a)$.
- For integer valued random variables: $f(k) = F(k) - F(k-1)$.

Expected value

We are often interested in the "average" outcome of a random variable.

Expected value

The expected value of a random variable is defined as

$$\mathsf{E}[X] = \sum_{\text{all } k} k f_X(k) \qquad \text{if } X \text{ is discrete,}$$

Recall: the average using fractions

Data set: grades of 24 students

$$5, 5, 6, 5, 6, 6, 6, 5, 5, 7, 6, 7, 5, 5, 5, 6, 6, 6, 5, 6, 5, 7, 6, 7$$

Table:

grade	$x_1 = 7$	$x_2 = 6$	$x_3 = 5$
fraction of students	$p_1 = 4/24$	$p_2 = 10/24$	$p_3 = 10/24$

Average One can write the average in different forms

$$\text{Average} = \frac{5 + 5 + 6 + \cdots + 5 + 7 + 6 + 7}{24}$$

$$= \frac{7 \cdot 4 + 6 \cdot 10 + 5 \cdot 10}{24} = 7 \cdot \frac{4}{24} + 6 \cdot \frac{10}{24} + 5 \cdot \frac{10}{24} = \sum_{i=1}^{3} x_i \cdot p_i$$

Expected value

The expected value of a discrete random variable X with finitely many outcomes can also be written as

$$\mu = \mathsf{E}[X] = \sum_{\text{all } k} x_k \cdot \underbrace{\mathsf{P}(X = x_k)}_{f(x_k)}$$

$$= x_1 \cdot \mathsf{P}(X = x_1) + x_2\, \mathsf{P}(X = x_2) + \cdots + x_n \cdot \mathsf{P}(X = x_n)$$

Here x_i are the n possible outcomes and $\mathsf{P}(X = x_i)$ are the probabilities of each outcome.

Expected value

Flip two coins... count the number of heads.

$f(0) = \frac{1}{4}, f(1) = \frac{1}{2}$ and $f(2) = \frac{1}{4}$

$\mathsf{E}[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$

Rules for computing expected values

For the expected value,

- $\mathsf{E}[a] = a$.
- $\mathsf{E}[aX] = a\mathsf{E}[X]$.
- $\mathsf{E}[aX + b] = a\mathsf{E}[X] + b$.
- $\mathsf{E}[X + Y] = \mathsf{E}[X] + \mathsf{E}[Y]$.

Here X and Y are any two random variables and a and b are constants.

Transformations

If we transform the random variables by a function h we have:

Theorem ♡

$$\mathsf{E}[h(X)] = \sum_{\text{all } k} h(k) f(k)$$

Coin example (with $h(x) = x/2$):

$$\mathsf{E}[X/2] = \frac{0}{2} \cdot f_X(0) + \frac{1}{2} \cdot f_X(1) + \frac{2}{2} \cdot f_X(2) = \tfrac{1}{2}$$

$$= (\mathsf{E}[X])/2$$

Common discrete distributions

Bernoulli distribution

The Bernoulli distribution describes a random experiment that can either succeed (with probability p) or fail (with probability $1 - p$.) Suppose we make a random experiment which succeeds with probability p and set

$$X = \begin{cases} 1, & \text{if the experiment succeeds} \\ 0, & \text{in case of failure.} \end{cases}$$

We have $f(1) = p$ and $f(0) = 1 - p$.

Sometimes useful to write as $f(k) = p^k(1 - p)^{1-k}$ for $k \in \{0, 1\}$.

The binomial distribution

Bernoulli distribution

A random variable X is Bernoulli distributed if it has probability mass function $f(1) = p$ and $f(0) = 1 - p$ and $= 0$ otherwise. We write $X \sim \mathrm{Ber}(p)$.

Expected value: $\mathsf{E}[X] = p$.

Examples?

The binomial distribution

The binomial distribution describes the probability of having exactly k successes in n independent Bernoulli trials with probability of success p.

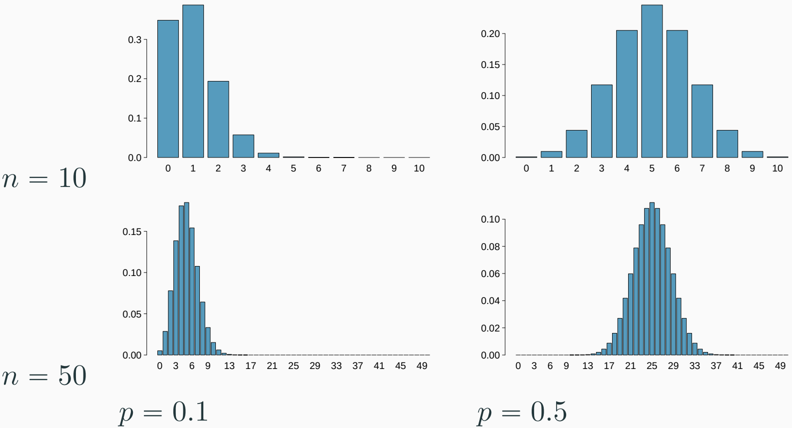
If X is binomial with parameters n and p we write:

$$X \sim \mathrm{Bin}(n, p)$$

Expected value: $\mathsf{E}[X] = np$.

Ha, the sum of two coins with sides 0 and 1 is $\mathrm{Bin}(2, 0.5)$ distributed.

The binomial distribution



The binomial distribution

The binomial distribution describes the probability of having exactly k successes in n independent Bernoulli trials with probability of success p.

If X is binomial with parameters n and p we write:

$$X \sim \mathrm{Bin}(n, p)$$

Binomial distribution

A random variable X is binomial distributed with parameters n, p if

$$\mathsf{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Sum of binomial distributed random variables

Sum of binomial distributed random variables.

If $X_1 \sim \mathrm{Bin}(n, p)$ and $X_2 \sim \mathrm{Bin}(m, p)$ are independent, then $X_1 + X_2 \sim \mathrm{Bin}(m + n, p)$.

("Dropping m items, couting the broken ones, dropping n more items, counting the additional broken ones is the same as dropping $m + n$ items...")

Geometric distribution

The experiment consists of a series of independent Bernoulli trials with probability of success equal to p.

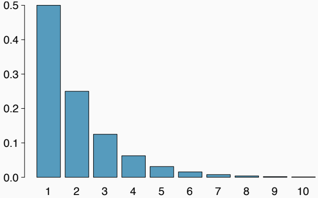
The random variable X denotes the number of trials needed to get the first success.

p is called the parameter of X.

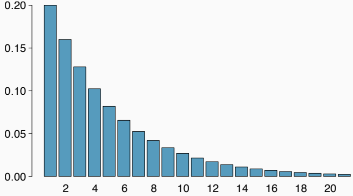
The geometric distribution

The geometric distribution describes the probability distribution of the number of trials needed k to get the first success, for a single event succeeding with probability p. ($k - 1$ failures and 1 success.)

$p = 0.5$



$p = 0.2$



The geometric distribution

Geometric distribution

A random variable X is geometrically distributed with parameters p if

$$\mathsf{P}(X = k) = (1-p)^{k-1}p, \quad k = 1, 2, \ldots$$

We write $X \sim \mathrm{Geom}(p)$.

Expected value: $\mathsf{E}[X] = \frac{1}{p}$.

Variance

Variance and standard deviation

Variance

The variance of a random variable is defined as

$$\mathsf{V}(X) = \mathsf{E}[(X-\mu)^2],$$

where $\mu = \mathsf{E}[X]$ is the expected value of X.

In words, this is the expected squared deviation of the mean. The variance can be calculated by

$$\mathsf{V}(X) = \sum_{\text{all } x} (x-\mu)^2 f(x)$$

Sometimes it is easiest to compute $\mathsf{V}(X) = \mathsf{E}[X^2] - \mu^2$.

The standard deviation of a random variable X is defined as $\sigma = \sqrt{\mathsf{V}(X)}$.

σ has the same units as X.

Rules for computing variance

For the variance

- $\mathsf{V}(a) = 0$.
- $\mathsf{V}(aX) = a^2\mathsf{V}(X)$.
- $\mathsf{V}(aX + b) = a^2\mathsf{V}(X)$.
- $\mathsf{V}(X + Y) = \mathsf{V}(X) + \mathsf{V}(Y)$, if X and Y are independent.

Here X and Y are two random variables and a and b are constants.