Lectures

MVE055 / MSG810
Mathematical statistics and discrete mathematics

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Last updated September 21, 2022

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Sum of Gaussian

Sum of Gaussian r.v.

Let $X \sim \mathrm{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathrm{N}(\mu_Y, \sigma_Y^2)$ with X and Y independent. Then

$$X + Y \sim \mathrm{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

Note: A normal random variable with mean μ and variance σ^2 has moment generating function $m(t) = \exp(t\mu + t^2\sigma^2/2)$. So if you tell me your moment generating function, I tell you if you are normally distributed and if, what your parameters are. We can prove the theorem by computing and identifying the m.g.f of $X + Y$ (next slide)

Proof with m.g.f.

So we now $m_X(t) = \mathsf{E}\exp(tX) = \exp(t\mu_X + t^2\sigma_X^2/2)$ and $m_Y(t) = \mathsf{E}\exp(tY) = \exp(t\mu_Y + t^2\sigma_Y^2/2)$.

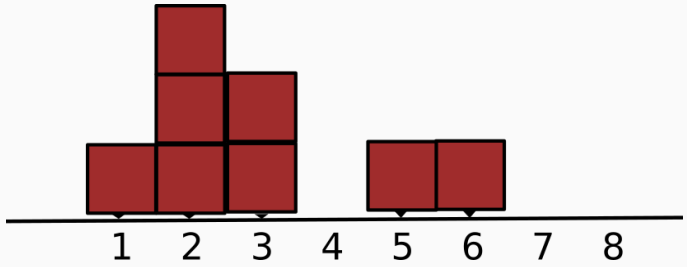
We compute and identify m_{X+Y}

$$\begin{aligned} m_{X+Y}(t) &= \mathsf{E}\exp(t(X+Y)) = \mathsf{E}\left(\exp(tX)\exp(tY)\right) \\ &\overset{indep}{=} \mathsf{E}\left(\exp(tX)\right)\mathsf{E}\left(\exp(tY)\right) \\ &= m_X(t)m_Y(t) = \exp(t\mu_X + t^2\sigma_X^2/2)\exp(t\mu_Y + t^2\sigma_Y^2/2) \\ &= \exp(t(\mu_X+\mu_Y) + t^2(\sigma_X^2+\sigma_Y^2)/2) \end{aligned}$$

which is m.g.f of $\mathrm{N}(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2)$ so $X+Y$ must be $\mathrm{N}(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2)$ distributed.

Data set

Case	1	2	3	4	5	6	7	8
Variable	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
Value	2	3	2	6	5	1	2	3



Weights on a bar

Sample median

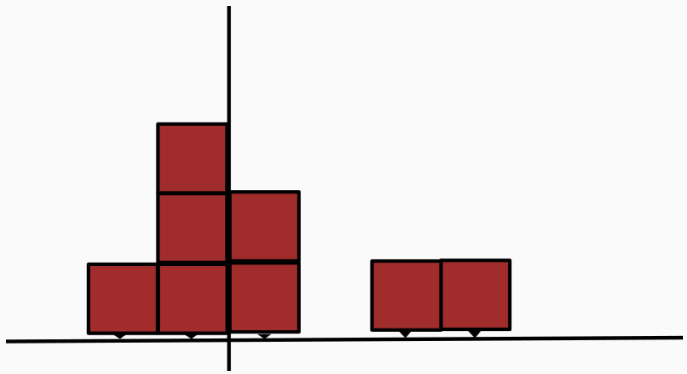
- To obtain the sample median,
  write the values in sorted order and take the middle one.

  If there is an even number of values in the data set, take the average of the two middle most.

Median

Median

Value	1	2	2	2	3	3	5	6

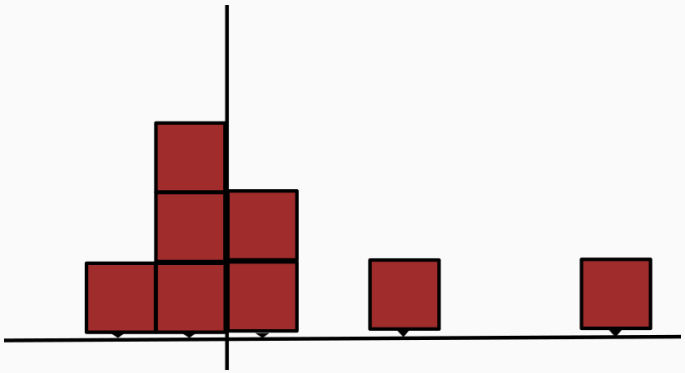


Sample median = 2.5

Sample median

Sample median

Value	1	2	2	2	3	3	5	8



Sample median = 2.5

Sample mean

- The (sample) mean, denoted as $\bar{x}$, can be calculated as

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

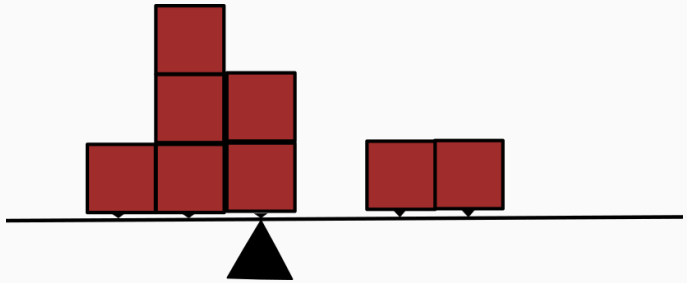
where $x_1, x_2, \cdots, x_n$ are the n observed values.

In words: Sum the values of all cases in the data set and divide by the total number of values.

Sample mean

Sample mean

Value	1	2	2	2	3	3	5	6

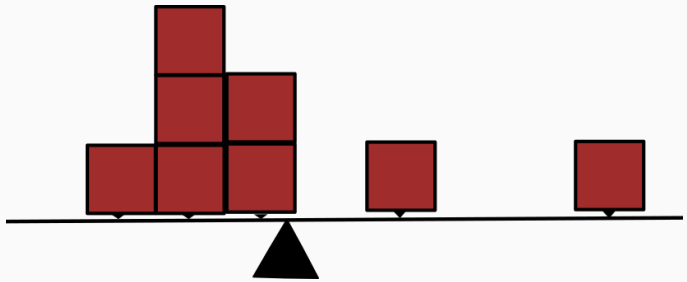


$$\text{Sample mean } \bar{x} = \frac{1\cdot 1 + 3\cdot 2 + 2\cdot 3 + 1\cdot 5 + 1\cdot 6}{8} = 3$$

Sample mean

Sample mean

Value	1	2	2	2	3	3	5	8



Sample mean $\bar{x} = \dfrac{1\cdot 1 + 3\cdot 2 + 2\cdot 3 + 1\cdot 5 + 1\cdot 8}{8} = 3.25$

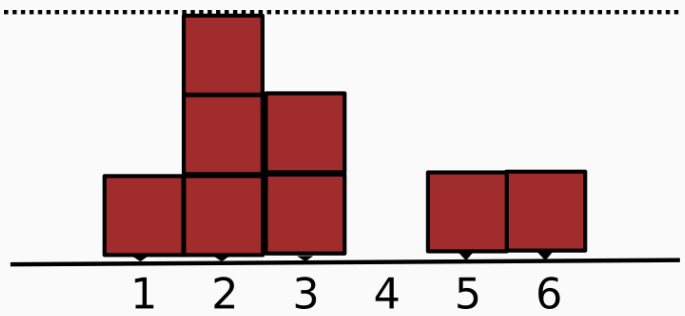
Sample mean, median and mode

- The sample mode is the value or label appearing most often.

Sample mean, median and mode

Sample mode

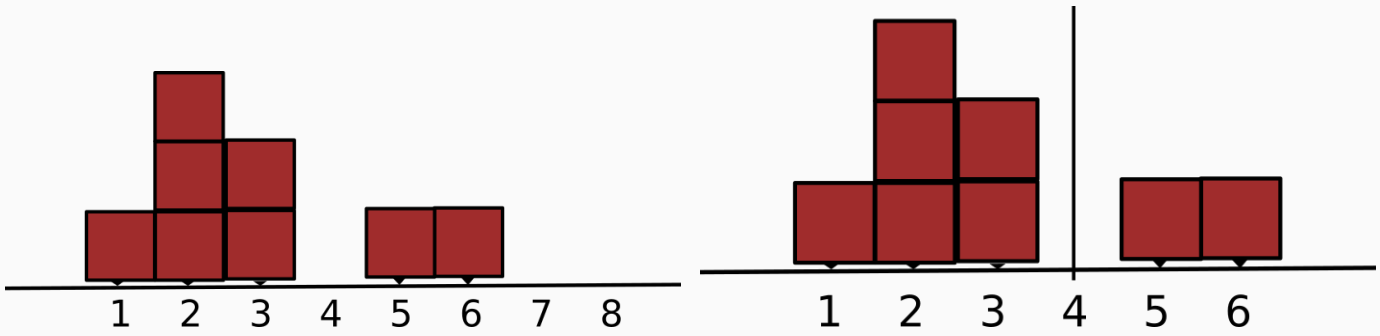
Value	1	2	2	2	3	3	5	6



Sample mode = 2

Empirical(Sample) quartiles

Case	1	2	3	4	5	6	7	8
Value	2	3	2	6	5	1	2	3



Recall: Half of the data is larger or equal the median, a half smaller or equal the median.

Empirical quartiles are the values of those cases that separate the ordered data into quarters.

Example: 3/4 of values are smaller than the 3rd quartile, 1/4 of values are larger than the 3 quartile.

Sample standard deviation

We call the squared distance of an observation from its mean its squared deviation.

For our example, as the mean $\bar{x} = 3$,

Case	1	2	3	4	5	6	7	8
value	2	3	2	6	5	1	2	3
Distance^2	$(2-3)^2$	$(3-3)^2$	$(2-3)^2$	$(6-3)^2$	$(5-3)^2$	$(1-3)^2$	$(2-3)^2$	$(3-3)^2$
	=1	=0	=1	=9	=4	=4	=1	=0

The standard deviation is the square root of the average of these values, but we take the average dividing by $n-1 (=7)$ instead of $n (=8)$

$$s = \sqrt{\frac{4 \cdot 1 + 1 \cdot 3 + 0 \cdot 2 + 4 \cdot 1 + 9 \cdot 1}{7}} \approx 1.69$$

Sample statistics for variation/spread

Sample variance: The sample variance of a data set $x_1, \ldots, x_n$ is given by

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n-1}((x_1 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2)$$

Sometimes convenient to use the formula

$$s^2 = \frac{1}{n-1}(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2) = \frac{1}{n-1}(x_1^2 + \ldots + x_n^2 - n\bar{x}^2)$$

Sample standard deviation s: the square root $\sqrt{s^2}$ of the sample variance.

Example 1 (cont.)

For the dice throw example

$$1, 3, 3, 3, 1, 6, 6, 5, 1, 4, 6, 1, 4, 5, 1, 1, 2, 3, 6, 5$$

we obtain the mean

$$\bar{x} = (1 + 3 + 3 + \ldots + 3 + 6 + 5)/20 = 67/20 = 3.35$$

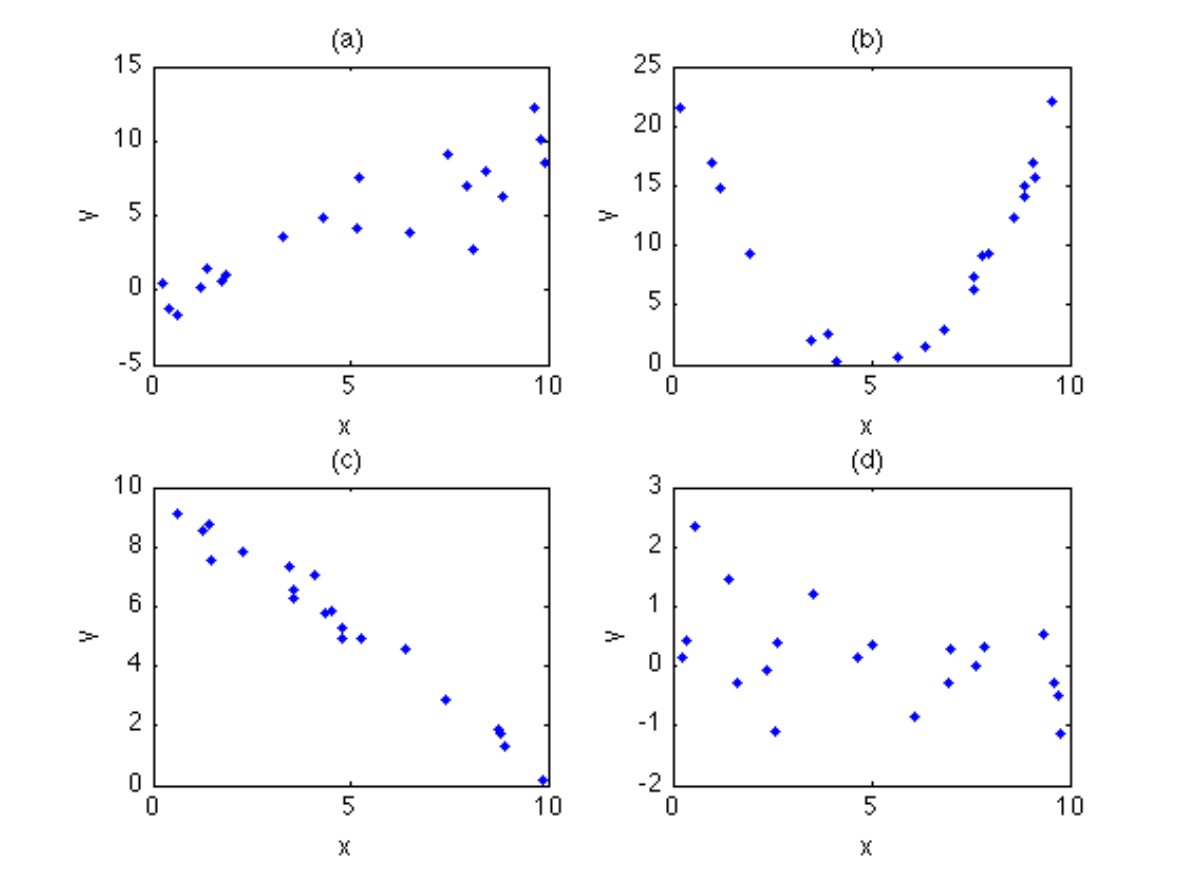
Sorting the values and taking the central one we obtain the median 3.

The variance is

$$s^2 = ((1 - 3.35)^2 + (3 - 3.35)^2 + \ldots + (5 - 3.35)^2)/19 = 3.8184$$

and the standard deviation is $s = 1.9541$.

Scatter plot of bivariate data



Assume 2d measurements (x_i, y_i). A scatter plot is a two-dimensional plot in which each (x_i, y_i) measurement is represented as a point in the x-y-plane.

Statistics for bivariate data

The *sample* covariance is defined as,

$$c_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

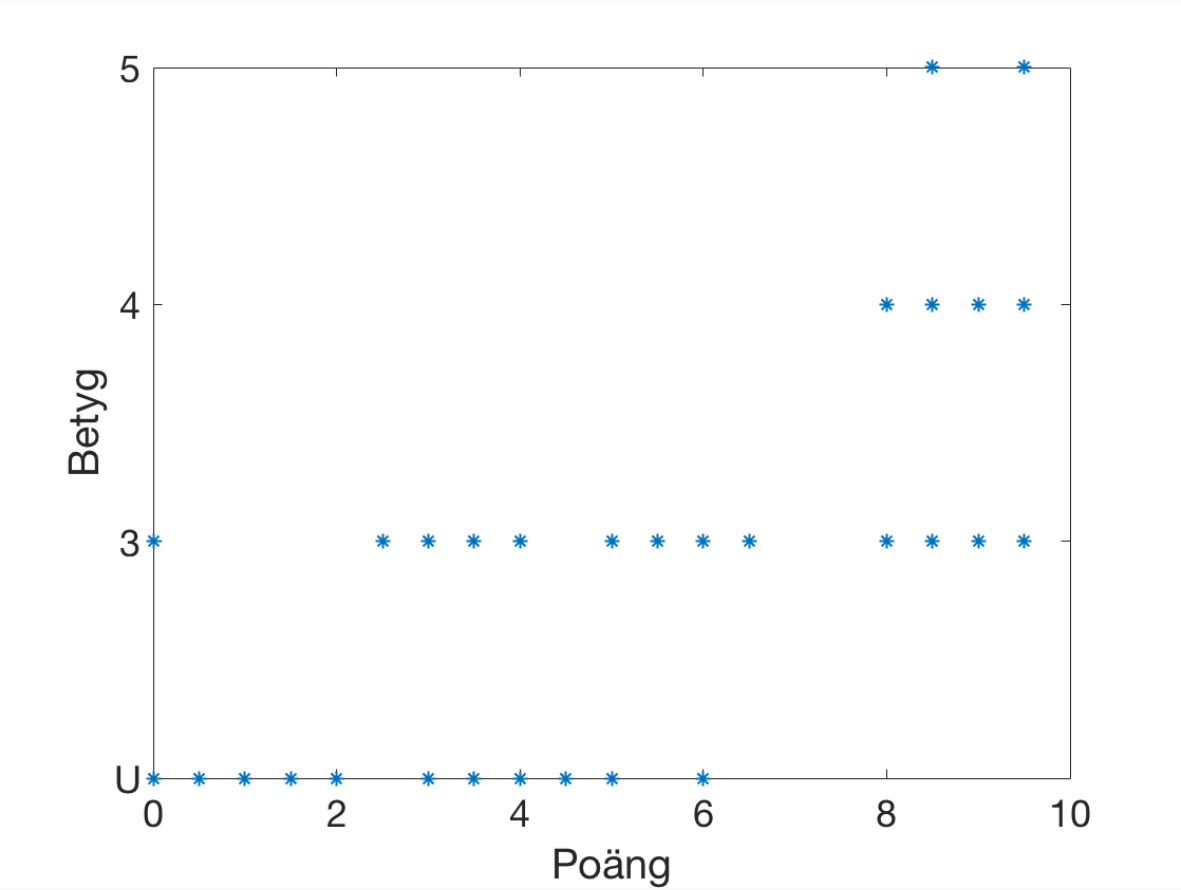
and is an unbiased estimator of the covariance $\mathrm{Cov}(X, Y)$.

The sample correlation coefficient is defined as

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}} = \frac{c_{xy}}{s_x s_y}$$

The sample correlation is an empirical measure of linear dependence.

Example: Course results 2017



Exam grade (Y) versus points in exam question 5 (X).
Correlation: $r_{xy} = 0.7261$

Samples and point estimators

Statistics

Example: $(5.27, 4.07, 5.48, 3.38)$ are measurements of the weight of $n = 4$ randomly (independent) selected cats.

The weight of a cat is modelled as normal random variable X_1, X_2, X_3, X_4 each $\mathrm{N}(\mu, (1.2)^2)$-distributed with unknown parameter μ. Here $\mathrm{N}(\mu, (1.2)^2)$ is a model for the population of *all cats*.

$(5.27, 4.07, 5.48, 3.38)$ is a sample of X_1, X_2, X_3, X_4.

Definition: Sample

A sample $(x_1, \ldots, x_n)$ of size n is made of n independent observations (realisations) of a random variable. Or – the same – of random variables $X_1, \ldots, X_n$ where all X_i are independent and equally distributed (thus have the same distribution).

Statistics

Example: $(5.27, 4.07, 5.48, 3.38)$ are measurements of the weight of $n = 4$ randomly (independent) selected cats.

The weight of a cat is modelled as normal random variable X_1, X_2, X_3, X_4 each $\mathrm{N}(\mu, (1.2)^2)$-distributed with unknown parameter μ. Here $\mathrm{N}(\mu, (1.2)^2)$ is a model for the population of *all cats*.

$(5.27, 4.07, 5.48, 3.38)$ is a sample of X_1, X_2, X_3, X_4.

Definition: Anti-Example

$(5.27, 5.27, 5.27, 5.27, 5.27)$ is perhaps not a sample

(lack of independence because some genius just weighted the same cat over and over).

Statistics

Like in the "cat"-example we can often say what kind of distribution is appropriate for X but we do not know the right parameters.

Many statistical problems can be reduced to the following question: Given the observations $x_1, \ldots, x_n$, what can we say about the parameters in the distribution of X_i (assuming each X_i is drawn independently from the same distribution)?

Definition: i.i.d.

We write $X_1, X_2, \ldots X_n \overset{\text{i.i.d.}}{\sim} D$ if $X_1, X_2, \ldots, X_n$ are independently and identically distributed with distribution D.

The sample mean as estimator

$\bar{X}^{(n)} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is the sample mean.

Example: Let $(5.27, 4.07, 5.48, 3.38)$ our sample.
$\bar{x}^{(4)} = (5.27 + 4.07 + 5.48 + 3.38)/4 = 4.55$ is a realisation $\bar{X}^{(n)}$.

We model $\bar{X}^{(n)}$ itself as random variable with its own expectation, variance and realization etc. Now with $\mu = \mathrm{E}[X_1] = \mathrm{E}(X_2) = ...$ and $\sigma^2 = \mathrm{V}(X_1) = \mathrm{V}(X_2) = ...$

$$\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[X_i] \overset{(*)}{=} \mu$$

$$\mathrm{V}\left(\frac{1}{n}\sum_{i=1}^{n} X_i)\right) \overset{i.i.d}{=} \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{V}(X_i) = \frac{\sigma^2}{n}$$

Ah! Smaller uncertainty, 4.55 is perhaps closer to μ than most the values in our sample which vary from μ by σ.

The sample mean as random variable

Expectation and variance of the sample average

$\mathrm{E}\left[\bar{X}^{(n)}\right] = \mu$ and $\mathrm{V}\left(\bar{X}^{(n)}\right) = \sigma^2/n$.

Quiz: How fast goes uncertainty down if n increases?

Standard error of the mean

$\frac{\sigma}{\sqrt{n}}$ is called standard error of the mean.

Point estimate and standard error

Example: Take $(5.27, 4.07, 5.48, 3.38)$ our sample. Model $X_1, \dots X_n \overset{i.i.d}{\sim} \mathrm{N}(\mu, \sigma^2)$ with $n = 4$ and $\sigma = 1.2$ and μ unknown.

$\bar{x}^{(4)} = (5.27 + 4.07 + 5.48 + 3.38)/4 = 4.55$ is an estimate for μ

The standard error associated with $\bar{x}^{(4)}$ is $\sigma/\sqrt{n} = 1.2/\sqrt{4} = 0.6$.

Our estimate

$$\mu \approx 4.55 \pm 0.6$$

The sample mean as random variable: Gaussian case

Average of Gaussian distributed random variables.

Let $X_1, \ldots, X_n$ an independent sample of a $\mathrm{N}(\mu, \sigma^2)$ r.v. Then $\bar{X}^{(n)}$ is $\mathsf{N}(\mu, \sigma^2/n)$-distributed.

Point estimators

Estimation

An estimator for a parameter θ is a function $\hat{\theta}(X_1, \ldots, X_n)$ mapping the observations into the parameter space Θ.

Example: $\bar{X}^{(n)}$ is an estimator for $\mu = \mathsf{E}[X_1] = \mathsf{E}[X_2] = ...$

$\hat{\theta}$ can refer both to a random variable and to actual observed values.

- $\hat{\theta}(X_1, \ldots, X_n)$ is a random variable with a certain distribution (random in $\rightarrow$ random out).
- $\hat{\theta}(x_1, \ldots, x_n)$ is a number calculated from data. This is called the point estimate of the parameter.

Properties of estimators

Two important qualities of estimators:

- *unbiased*: $\mathsf{E}[\hat{\theta}(X_1, \ldots, X_n)] = \theta$.
- Small variance in large samples: $\mathsf{V}(\hat{\theta}(X_1, \ldots, X_n))$ small if n large.

If the expected value of the estimator is the true value (the estimator is unbiased), that means that the estimated values center on average around the true value if we make several repeated samples of size n.

- For a given sample, the value need not be close to the true value.
- The standard deviation of an unbiased estimate gives an indication of how far it may be from the actual value.
- Often the standard error of the estimate is reported, which is the standard deviation of the estimate.

Important estimators

Sample mean and sample variance

Consider an i.i.d sample $(X_1, \ldots, X_n)$ and assume that $\mathsf{E}[X_i] = \mu$ and $\mathsf{V}(X_i) = \sigma^2$.

The **sample mean** $\hat{\mu} = \bar{X}^{(n)}$ is an unbiased estimator of μ, that is $\mathsf{E}[\hat{\mu}] = \mu$. It has standard error $\sqrt{V(\hat{\mu})} = \frac{\sigma}{\sqrt{n}}$.

An unbiased estimator for the variance σ^2 is the **sample variance**

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

Sample variance can also be computed as

$$S^2 = \frac{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}{n(n-1)}$$

Percentiles and quantiles

The p^{th} percentile P is the value of X such that $p\%$ or less of the observations are less than P and $(100-p)\%$ or less are greater than P. p^{th} percentiles are $p\%$-quantiles.

In particular, P_{25} is the 25^{th} percentile or the first quartile denoted also by Q_1.P_{50} is the 50^{th} percentile or the second quartile Q_2, which is also the median, and P_{75} is the 75^{th} percentile or the third quartile Q_3.

Note that $Q_1 = \frac{n+1}{4}$ th ordered observation, $Q_2 = \frac{2(n+1)}{4} = \frac{n+1}{2}$ th ordered observation, and $Q_3 = \frac{3(n+1)}{4}$ th ordered observation.

Example

Given the following set of data :

$$18, 1, 20, 15, 12, 15, 14, 7, 11, 9, 6, 4$$

Order the numbers from the lowest to the highest

$$1, 4, 6, 7, 9, 11, 12, 14, 15, 15, 18, 20$$

$\bar{x}^{(12)} = \frac{1+4+\cdots+18+20}{12} = 11.$

Median: $\mathrm{Me} = \frac{11+12}{2} = 11.5.$

Example

Given the following set of data :

$$18, 1, 20, 15, 12, 15, 14, 7, 11, 9, 6, 4$$

Variance

$$s^2 = \frac{(20-11)^2 + (18-11)^2 + \cdots + (-7)^2 + (-10)^2}{12-1} \approx 33.3$$

Order the numbers from the lowest to the highest

$$1, 4, 6, 7, 9, 11, 12, 14, 15, 15, 18, 20$$

$Q_1 = 6.25$, $Q_3 = 15$.

Boxplot