Recall: Piazza

$$\text{(BVP)} \quad \begin{cases} -u''(x) = f(x) & 0 < x < 1 \\ u(0) = 0, \; u(1) = 0 \end{cases}$$

- Variational formulation:

test fct

(VF) Find $u \in V^0$ s.t. $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in V^0$,

$$V^0 = \{ v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1), \; v(0) = v(1) = 0 \}$$

$$(u', v')_{L^2} = \int_0^1 u'(x) v'(x) \, dx$$

- FE problem:

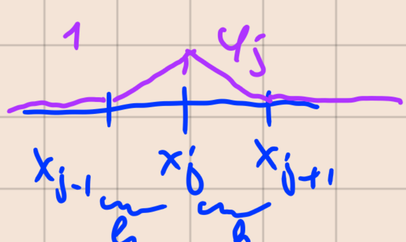
"chi"

(FE) Find $U \in V_h^0$ s.t. $(U', \chi')_{L^2} = (f, \chi)_{L^2} \quad \forall \chi \in V_h^0$

$$V_h^0 = \{ v : [0,1] \to \mathbb{R} : v \text{ cont. pw. linear on } T_h, \; v(0) = v(1) = 0 \}$$

$$= \operatorname{span}(\varphi_1, \varphi_2, \dots, \varphi_m) \subset V^0, \quad \dim(V_h^0) = m.$$

$\varphi_j \rightsquigarrow$ hat fct



T_h: uniform partition of $[0,1]$: $0 = x_0 < x_1 < x_2 < \dots < x_{m+1} = 1$
with $x_j - x_{j-1} = h$

To find a linear system of eq. from (FE), we write U as

$$U(x) = \sum_{j=1}^{m} \underbrace{\xi_j}_{\text{coordinates (?)}} \underbrace{\varphi_j(x)}_{\text{basis}}$$

(since $U \in V_h^0 = \text{span}(\varphi_1, \dots, \varphi_m)$)

Further, it is <u>enough</u> to take $\chi = \varphi_i$ for $i = 1, 2, 3, \dots, m$ (φ_i build a basis of V_h^0)

We insert the above into (FE) and get:

$(U', \chi') = (f, \chi)$

(FE) Find ξ_j s.t. $\left(\sum_{j=1}^{m} \xi_j \varphi_j', \varphi_i'\right)_{L^2} = (f, \varphi_i)$ for $i = 1, 2, \dots, m$.

$$\Leftrightarrow \quad \sum_{j=1}^{m} \xi_j \, (\varphi_j', \varphi_i')_{L^2} = (f, \varphi_i)_{L^2} \quad \forall i = 1, \dots, m.$$

$$\Leftrightarrow \quad \sum_{j=1}^{m} \xi_j \underbrace{\int_0^1 \varphi_j'(x) \varphi_i'(x) \, dx}_{a_{ij}} = \underbrace{\int_0^1 f(x) \varphi_i(x) \, dx}_{b_i} \quad i = 1, \dots, m.$$

$(\Leftrightarrow)$ $\sum_{j=1}^{m} a_{ij} \xi_j = b_i$ for $i=1,\dots,m$

$(\Leftrightarrow)$ $A \cdot \vec{\xi} = \vec{b}$ a linear syst. of eq.

matrix vector

vector (?)

4) (4th. step of Galerkin)

If we can compute A and $\vec{b}$, we get $\vec{\xi}$ and hence u, the sol. to (FE).

The matrix A is called the stiffness matrix and has entries given by

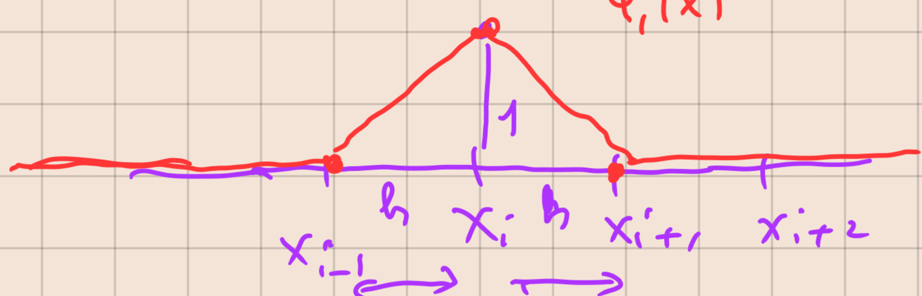
$$a_{ij} = \int_0^1 \varphi_i'(x) \cdot \varphi_j'(x)\,dx \,, \quad i=1,\dots,m, \; j=1,\dots,m$$

Details for a_{ii}: for $i = 1, \dots, m$

$$a_{ii} \overset{\text{Def}}{=} \int_0^1 \varphi_i'(x)\cdot\varphi_i'(x)\,dx = \int_0^1 \left(\varphi_i'(x)\right)^2 dx =$$

$$= \int_{0=x_0}^{x_1} (\varphi_i'(x))^2\,dx + \int_{x_1}^{x_2} (\varphi_i'(x))^2\,dx + \dots + \int_{x_{i-1}}^{x_i} (\varphi_i'(x))^2\,dx$$

$$+ \int_{x_i}^{x_{i+1}} (\varphi_i'(x))^2\,dx + \dots + \int_{x_m}^{1=x_{m+1}} (\varphi_i'(x))^2\,dx =$$



$$\overset{\text{Def } \varphi_i}{=} 0 + 0 + 0 + \dots + \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(\frac{1}{h}\right)^2 dx + 0 + \dots + 0$$

$$= \frac{1}{h^2} \underbrace{(x_i - x_{i-1})}_{h} + \frac{1}{h^2} \underbrace{(x_{i+1} - x_i)}_{h} = \frac{2}{h} \quad //$$

Similarly, we can compute

$a_{i,i+1}$, $a_{i-1,i}$ and a_{ij} for $|i-j| > 2$

and get

zero everywhere else

$$A = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & -1 & 2 \end{pmatrix}$$

a_{ii}

$a_{i,i+1}$

$a_{i-1,i}$

a_{ij}

Next, we compute the load vector $\vec{b}$, which has entries

$$b_i = \int_0^1 f(x)\varphi_i(x)\,dx$$

for $i = 1, 2, \dots, m$.

One can compute these integrals exactly if $f \cdot \varphi_i$ is "easy", else we have to use a numerical integration! (next chapt.)

Chapter IV: Interpolation and numerical integration

Goal: Interpol: Pass a (simple) function though data points

Num. integration: Find numerical approximation to

$$\int_a^b f(x)\,dx \approx ??$$

1) Polynomial interpolation:

Ideas / Def:

Consider $f: [a,b] \to \mathbb{R}$ a continuous fct.

and $(q+1)$ distinct points

$$a = x_0 < x_1 < x_2 \dots < x_{q-1} < x_q = b$$

We call the data $(x_j, f(x_j))_{j=0}^{q}$

interpolation points.

A polyn. $\Pi_q f \in \mathcal{P}^{(q)}(a,b)$ is called a polynomial interpolant if

$$\Pi_q f(x_j) = f(x_j) \quad \text{for } j = 0, 1, \ldots, q.$$

Ex: (linear interpolation on $[0,1]$, classical basis)

- linear $\Rightarrow$ $q=1$ $\Rightarrow$ $0 = x_0$ and $1 = x_1$.
- We know that $\mathcal{P}^{(1)}(0,1) = \text{span}(1, x)$ $\Rightarrow$
  (polyn. def. on $[0,1]$)

$$\Pi_1 f(x) = a_0 \cdot 1 + a_1 \cdot x$$

coordinates (unknown)

basis

- We need 2 conditions to find a_0 and a_1:

$$\Pi_1 f(0) \overset{!}{=} f(0) \Leftrightarrow a_0 + a_1 \cdot 0 \overset{!}{=} f(0) \Rightarrow a_0 = f(0)$$

$$\Pi_1 f(1) \overset{!}{=} f(1) \Leftrightarrow a_0 + a_1 \cdot 1 = f(1) \Rightarrow a_1 = f(1) - f(0)$$

$$\hookrightarrow \Pi_1 f(x) = \underbrace{f(0)}_{a_0} + \underbrace{(f(1) - f(0))}_{a_1} \cdot x \qquad \text{[eq. line]}$$

Ex1 (linear interpolation on [0,1], Lagrange basis)

We do the same as above excepte that

- We consider $\mathbb{P}^{(1)}(0,1) = \text{span}(\lambda_0(x), \lambda_1(x))$,

where the Lagrange polyn. are given by

$$\lambda_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x-1}{0-1} = 1-x$$

$$\lambda_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x-0}{1-0} = x$$

see previous chapter

- We can thus write $\Pi_1 f \in \mathbb{P}^{(1)}(0,1)$ as

$$\Pi_1 f(x) = b_0 \cdot \lambda_0(x) + b_1 \cdot \lambda_1(x) = b_0(1-x) + b_1 \cdot x$$

coordinates

basis

- We find b_0 and b_1 using

!we want

$$\Pi_1 f(0) = f(0) \iff b_0 + b_1 \cdot 0 \overset{!}{=} f(0) \Rightarrow b_0 = f(0)$$

$$\Pi_1 f(1) = f(1) \iff b_1 = f(1)$$

$\hookrightarrow \Pi_1 f(x) = f(0)\cdot(1-x) + f(1)\cdot x$ //

? What is the error of the interpolant $\Pi_q f$?

Recall

$$\|f\|_{L^p(a,b)} = \left(\int_a^b |f(x)|^p dx \right)^{1/p} \qquad (p = 1,2)$$

$$\|f\|_{L^\infty(a,b)} = \max_{a \le x \le b} |f(x)|$$

$\hookrightarrow$ The "distance" between 2 fct is given by $\|f-g\|_{L^p}$ or $\|f-g\|_{L^\infty}$

Proposition: Let $p = 1,2,\infty$. Assume $f'' \in L^p(a,b)$.

Then, $\exists$ constants $C_1, C_2, C_3\ (>0)$ s.t.

(i) $\|\Pi_1 f - f\|_{L^p(a,b)} \le C_1 (b-a)^2 \|f''\|_{L^p(a,b)}$

(ii) $\|\Pi_1 f - f\|_{L^p(a,b)} \le C_2 \cdot (b-a) \|f'\|_{L^p(a,b)}$

(iii) $\|(\Pi_1 f)' - f'\|_{L^p(a,b)} \le C_3 (b-a) \|f''\|_{L^p(a,b)}$

Proof: (of (i) and $p = \infty$)

- From the example above, we know that

$$\Pi_1 f(x) = f(a) \cdot \lambda_0(x) + f(b) \cdot \lambda_1(x)$$

- Taylor expansions for $f(a)$ and $f(b)$:

$$f(a) = f(x) + (a-x) f'(x) + \frac{(a-x)^2}{2} f''(\xi_0) \quad \text{for } \xi_0 \in (a,x)$$

$$f(b) = f(x) + (b-x) f'(x) + \frac{(b-x)^2}{2} f''(\xi_1) \quad \text{for } \xi_1 \in (x,b)$$

- Insert above in $\Pi_1 f(x)$:

$$\Pi_1 f(x) = f(a)\lambda_0(x) + f(b)\lambda_1(x) = \dots =$$

$$= f(x) + \frac{1}{2}(a-x)^2 f''(\xi_0)\lambda_0(x) + \frac{1}{2}(b-x)^2 f''(\xi_1)\lambda_1(x)$$

$$\Rightarrow \quad |\Pi_1 f(x) - f(x)| \leq \frac{1}{2}|a-x|^2 |f''(\xi_0)| |\lambda_0(x)| +$$

$$+ \frac{1}{2}|b-x|^2 |f''(\xi_1)| \cdot |\lambda_1(x)|$$