Recall:

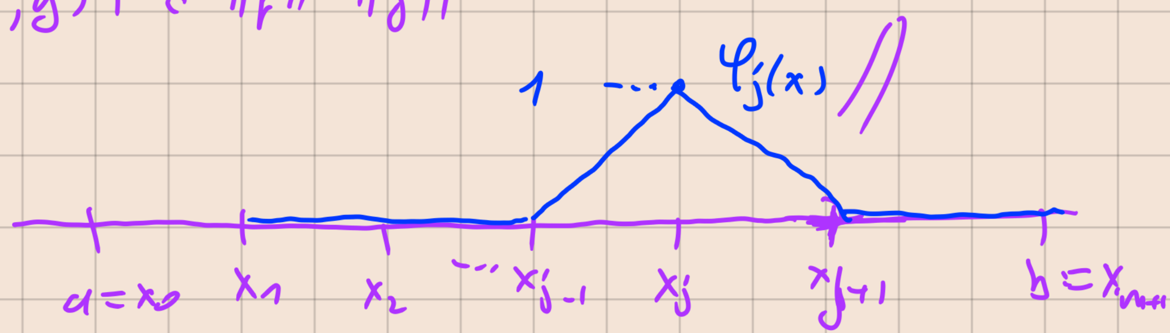
$$L^2(a,b) = \left\{ f : [a,b] \to \mathbb{R} : \int_a^b f(x)^2 dx < \infty \right\}$$

$$(f,g)_{L^2} = (f,g)_{L^2(a,b)} = \int_a^b f(x) g(x) \, dx$$

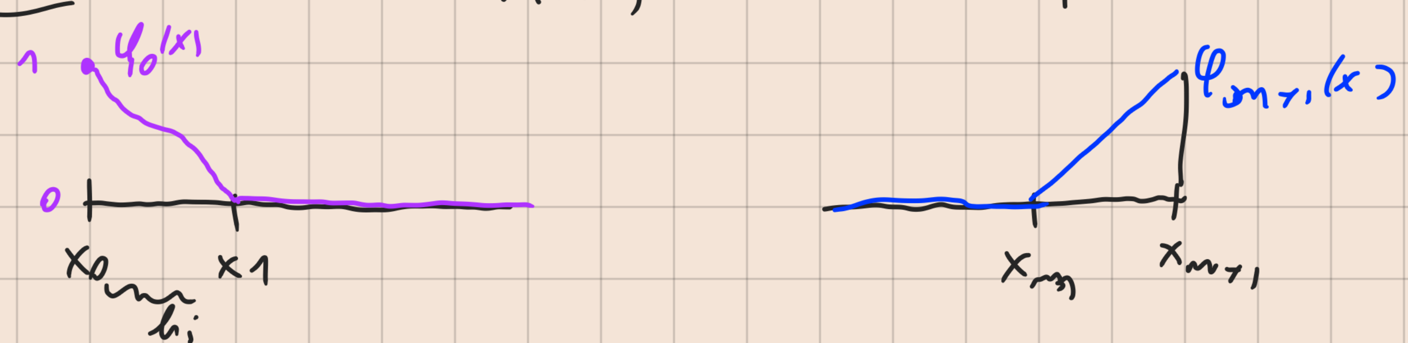
$$\|f\|_{L^2} = \sqrt{\int_a^b f(x)^2 dx}$$

C-S : $|(f,g)| \leq \|f\| \cdot \|g\|$

Hat fct



Rem: At x_0 and x_{m+1}, we have half hat fct.



Def: The space of continuous piecewise linear (pw) functions on $[0,1]$ is defined as

$$V_h([0,1]) = V_h = \text{span}(\varphi_0, \varphi_1, \ldots, \varphi_{m+1})$$

hat fct.

Rem: Any $v \in V_h$ can be written as

$$v(x) = \sum_{j=0}^{m+1} \xi_j \cdot \underbrace{\varphi_j(x)}, \text{ where } \xi_j = v(x_j) \ (*) \quad (\varphi_j(x_j) = 1)$$

coordinates $\quad$ hat functions "basis"

$$(*) \ v(x_i) = \sum_{j=0}^{m+1} \xi_j \underbrace{\varphi_j(x_i)}_{\substack{=0 \text{ for } i \neq j \\ =1 \text{ for } i=j}} = \xi_i \underbrace{\varphi_i(x_i)}_{=1} = \xi_i$$

$$\Downarrow$$

$$\xi_i = v(x_i)$$

2) L^2-projection:

Def: Let $q \in \mathbb{N}$, $f \in L^2(a,b)$. The L^2-projection of f is the polynomial $Pf \in \mathcal{P}^{(q)}(a,b)$ s.t.

$$(f, p)_{L^2} = (Pf, p)_{L^2} \quad \forall \ p \in \mathcal{P}^{(q)}(a,b)$$

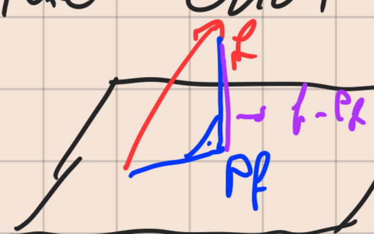
or

$$(*) \quad \int_a^b f(x)\,p(x)\,dx = \int_a^b Pf(x)\,p(x)\,dx \quad \forall p \in \mathcal{P}^{(q)}(a,b)$$

or

$$(f - Pf, p)_{L^2} = 0 \qquad \forall p \in \mathcal{P}^{(q)}(a,b).$$

Rem: The last eq., $(f-Pf,p)_{L^2} = 0$, says that the error $f - Pf \perp \mathcal{P}^{(q)}(a,b)$



Rem: Since $\mathcal{P}^{(q)}(a,b) = \text{span}(1, x, x^2, \dots, x^q)$, it is enough to consider $p(x) = x^j$ for $j = 0, 1, 2, \dots, q$ in $(*)$:

$$\int_a^b f(x)\,x^j\,dx = \int_a^b Pf(x)\,x^j\,dx \quad \text{for } j = 0,1,2,\dots,q.$$

Ex: Let $f \in L^2(a,b)$, what is $Pf \in \mathcal{P}^{(0)}(a,b)$? ($0 = q$)

Since $q=0$, the above tells us to consider

$$\int_a^b f(x)\cdot x^j\,dx = \int_a^b Pf(x)\cdot x^j\,dx \quad \text{for } j=0$$

$x^0 = 1$

$$\Longleftrightarrow \int_a^b f(x)\,dx = \int_a^b \underbrace{Pf(x)}_{\substack{\in \mathcal{P}^{(0)}(a,b) = \{\text{polyn. of degree} \le 0\} \\ \text{(by def)}}}\,dx$$

$= \{\text{constant polynomials}\}$

$$\Longleftrightarrow \int_a^b f(x)\,dx = \underbrace{Pf(x)}_{\text{constant}}\cdot \int_a^b dx = Pf(x)\cdot(b-a)$$

$$\Rightarrow Pf(x) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

<u>Rem</u>: The L^2-projection Pf is a kind of <u>averaged</u> approximation of $f \in L^2(a,b)$

Proposition:

(i) The L^2-projection is unique

(ii) The L^2-projection $Pf \in \mathcal{P}^{(q)}(a,b)$ is the best approximation of $f \in L^2(a,b)$ in the L^2-norm:

$$\underbrace{\| f - Pf \|_{L^2}}_{\text{error in } L^2\text{-proj.}} \leq \| f - p \|_{L^2} \quad \forall p \in \mathcal{P}^{(q)}(a,b)$$

Proof:

(i) Assume we have 2 L^2-projections called Pf_1 and Pf_2. By def,

$$(f, p)_{L^2} = (Pf_1, p)_{L^2} \quad \forall p \in \mathcal{P}^{(q)}(a,b)$$

$$(f, p)_{L^2} = (Pf_2, p)_{L^2} \quad \forall p \in \mathcal{P}^{(q)}(a,b).$$

Difference:

$$0 = (Pf_1 - Pf_2, p)_{L^2} \quad \forall p \in \mathcal{P}^{(q)}(a,b).$$

Take $\underset{\in \mathcal{P}^{(q)}(a,b)}{p} = \underset{\in \mathcal{P}^{(q)}(a,b)}{Pf_1} - \underset{\in \mathcal{P}^{(q)}(a,b)}{Pf_2}$ above: $0 = (Pf_1 - Pf_2, Pf_1 - Pf_2)_{L^2}$

Same term

This is just $0 \overset{}{\underset{\text{Def. of } L^2\text{-norm}}{=}} \|Pf_1 - Pf_2\|_{L^2}^2 \Rightarrow Pf_1 - Pf_2 = 0$

$$\Downarrow$$

$Pf_1 = Pf_2$!!

(ii) Let $p \in \mathcal{P}^{(q)}(a,b)$ and consider

$$\|f - Pf\|_{L^2}^2 \underset{\text{Def norm}}{=} (f - Pf, f - p + p - Pf)_{L^2} =$$

$$= (f - Pf, f - p)_{L^2} + (f - Pf, \overbrace{\underset{\in \mathcal{P}^{(q)}(a,b)}{p} - \underset{\in \mathcal{P}^{(q)}(a,b)}{Pf}}^{\in \mathcal{P}^{(q)}(a,b)})_{L^2} =$$

$$= (f - Pf, f - p)_{L^2} + 0 \underset{f - Pf \perp \mathcal{P}^{(q)}(a,b)}{=} \quad ⚠$$

$$= (f - Pf, f - p)_{L^2} \overset{C-S}{\leq} \|f - Pf\|_{L^2} \cdot \|f - p\|_{L^2}$$

$$\Rightarrow \|f - Pf\|_{L^2} \leq \|f - p\|_{L^2} \quad \forall p \in \mathcal{P}^{(q)}(a,b) \quad :-)$$

3) Galerkin method for IVP:

Look at home (Book / compendium)

4) Galerkin Finite Element for BVP:

Prob:

$$(BVP) \begin{cases} -u''(x) = f(x) \quad \text{for } 0 < x < 1 & \text{DE} \\ u(0) = 0 \text{ and } u(1) = 0 & \text{BC} \end{cases}$$

where f is given.

Rem: The above boundary conditions (BC) are called homogeneous Dirichlet BC.

Main steps for Galerkin approximation:

We find an approximation of the sol. to (BVP) using the following steps:

(i) Multiply DE with a test function v

(ii) Integrate the above over $(0,1)$ to get a variational formulation of the problem (VF)

(iii) Do a Finite Element Method (FEM):

$u \approx U$ a continuous pw linear fct (= piecewise)

(iv) Solve a linear system of equations to find U.

Details:

(i) Consider the space of test functions

$$V^0 := \{ v : [0,1] \to \mathbb{R} \;:\; v, v' \in L^2(0,1) \text{ and } \underline{v(0) = v(1) = 0} \}$$

Multiply DE with $v \in V^0$:

$$-u''(x) \cdot v(x) = f(x) \cdot v(x)$$

(ii) Integrate the above:

$$\underbrace{-\int_0^1 u''(x) \cdot v(x)\,dx}_{\text{by part}} = \int_0^1 f(x)\,v(x)\,dx \qquad \forall v \in V^0$$

$$\Leftrightarrow \underbrace{-u'(x)v(x)\Big|_0^1}_{-u'(1)\underbrace{v(1)}_{=0}+u'(0)\underbrace{v(0)}_{=0}\ \text{since } v\in V^0} + \int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v\in V^0$$

We thus get the variational formulation

(VF) Find $u \in V^0$ s.t. $\displaystyle\int_0^1 u'(x)v'(x)\,dx = \int_0^1 f(x)v(x)\,dx \quad \forall v\in V^0$

or

(VF) Find $u\in V^0$ s.t. $(u',v')_{L^2} = (f,v)_{L^2} \quad \forall v \in V^0$

(iii) V^0 is a BIG SPACE $\rightsquigarrow$ (VF) is difficult.

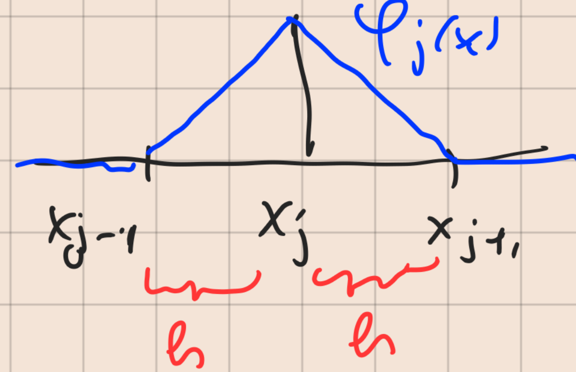
But we can work in a smaller space...

In order to do a Finite Element Method (FEM) we consider the space $V_h^0 \subset V^0$ defined

$$V_h^0 = \{ v:[0,1]\to\mathbb{R} : v \text{ is cont. pw linear on } T_h \text{ and } v(0)=0=v(1)\} =$$

$= \text{span}(\varphi_1, \varphi_2, \ldots, \varphi_m)$, where

T_h partition (uniform) of $[0,1]$ and φ_j are the hat functions



The FE problem thus reads

"CHI"

(FE) Find $U \in V_h^0$ s.t. $(U', \chi') = (f, \chi)$

for $\chi \in V_h^0$

$\dim(V_h^0) = m$