2) Basis of a vector space:

Let V be a vector sp./linear sp.

Def: A set $\{v_1, \ldots, v_n\}$ in V is linearly independent if the equation $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ has only the trivial solution $a_1 = a_2 = \ldots = a_n = 0$. Else, it is called linearly dependent

Ex: Let $V = \mathcal{P}^{(2)}([1,3])$. Show that the polyn. $p_1(x) = 1$, $p_2(x) = x$, $p_3(x) = x^2$ are linearly independent:

Consider eq. $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = \underset{\text{polyn. zero!}}{0} \overset{\text{def } p_1, p_2, p_3}{\Leftrightarrow}$

$a_1 + a_2 x + a_3 x^2 = 0 \quad \forall x \in [1,3]$

The only possibility for the above to happen is when $a_1 = a_2 = a_3 = 0$! Hence p_1, p_2, p_3 are lin. indep. polyn. in $\mathcal{P}^{(2)}([1,3])$

Ex: Show that the polyn. $p_1(x) = 2 - x^2$, $p_2(x) = 3x$, $p_3(x) = x^2 + x - 2$ are lin. depend. in $\mathcal{P}^{(2)}(\mathbb{R})$.

Consider $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = 0 \ \forall x \in \mathbb{R}$, i.e.

$a_1(2 - x^2) + a_2 3x + a_3 (x^2 + x - 2) = 0$ or

$(a_3 - a_1) x^2 + (3a_2 + a_3) x + 2(a_1 - a_3) = 0$

Only possible if
$$\begin{cases} a_3 - a_1 = 0 & \rightarrow a_3 = a_1 \\ 3a_2 + a_3 = 0 & \rightarrow a_3 = -3a_2 \\ a_3 - a_1 = 0 & \rightarrow a_1 = a_3 \end{cases}$$

Take for expl. $a_2 = 1 \Rightarrow a_3 = -3 \rightarrow a_1 = -3$ and we see that the polyn. $\{p_1, p_2, p_3\}$ are lin. dep. in $\mathcal{P}^{(2)}(\mathbb{R})$.

Rem: Here, one can also see that one can express $p_2(x)$ in terms of $p_1(x)$ and $p_3(x)$: $p_2(x) = 3 \cdot (p_3(x) + p_1(x))$!! This is always the case for lin. dep. sets of vectors.

Def:
- A set $\{v_1, v_2, \ldots, v_n\}$ in V is called a basis of V if $\text{span}\{v_1, v_2, \ldots, v_n\} = V$ and the set is lin. independent.
- The dimension of V is then $\dim(V) = n =$ # elements of basis

Ex: The standard basis of $V=\mathbb{R}^2$ is given by $\{e_1,e_2\}=\{\binom{1}{0},\binom{0}{1}\}$:

(i) e_1 and e_2 are lin. indep: $a_1\vec{e}_1+a_2\vec{e}_2=\vec{0} \Leftrightarrow a_1\binom{1}{0}+a_2\binom{0}{1}=\binom{0}{0}$

$\Leftrightarrow \binom{a_1}{a_2}=\binom{0}{0} \Leftrightarrow a_1=a_2=0 \Rightarrow \{e_1,e_2\}$ lin. indep!

(ii) $\operatorname{span}\{e_1,e_2\} \underset{\text{def span}}{=} \{a_1e_1+a_2e_2 : a_1,a_2\in\mathbb{R}\}=\left\{\binom{a_1}{a_2} : a_1,a_2\in\mathbb{R}\right\} \underset{\text{def } \mathbb{R}^2}{=} \mathbb{R}^2$!

(iii) $\#\operatorname{span}\{e_1,e_2\}=2 \Rightarrow \dim(V)=\dim(\mathbb{R}^2)=2$!

Ex: Show that $\{1,x,x^2\}$ is a basis of $V=\mathcal{P}^{(2)}(\mathbb{R})$:

(i) $\{1,x,x^2\}$ lin. indep? Ok, see above

(ii) $\operatorname{span}\{1,x,x^2\}=\mathcal{P}^{(2)}(\mathbb{R})$? Ok, see above

(iii) $\#\operatorname{span}(1,x,x^2)=3=\dim(\mathcal{P}^{(2)}(\mathbb{R}))$;-)

3) Inner product:

Rem: Vectors in $\mathbb{R}^2$'s $(1,2)\cdot\binom{3}{4}=1\cdot 3+2\cdot 4=11$ $\rightsquigarrow$ same now but abstract

Let V be a vector space / linear space.

Def: A scalar product / inner product on V is a map $(\cdot,\cdot):V\times V\to\mathbb{R}$ s.t.

(i) $(u,v)=(v,u)$ (symmetry)

(ii) $(u+\alpha v,w)=(u,w)+\alpha(v,w)$ (linearity)

(iii) $(v,v)\geq 0$ (positivity)

(iv) $(v,v)=0 \Leftrightarrow v=0$

$\forall\, \alpha\in\mathbb{R},\ u,v,w\in V$.

Def: • A linear space with an innerproduct is called an inner product space. Notation: $(V,(\cdot,\cdot))$ or $(V,(\cdot,\cdot)_V)$ or $(V,\langle\cdot,\cdot\rangle)$

• Such a space has a norm defined by $\|u\|=\sqrt{(u,u)}$

Ex: • $V=\mathbb{R}^2$ with standard inner prod. is an inner prod. space:

$\forall u,v\in V$, one has $u^Tv=u\cdot v=(u,v)=\langle u,v\rangle=\left(\binom{u_1}{u_2},\binom{v_1}{v_2}\right)=u_1v_1+u_2v_2$

• Similarly $V=\mathbb{R}^n$ with $(u,v)=\sum_{i=1}^n u_iv_i$ and norm $\|u\|=\sqrt{\sum_{i=1}^n u_i^2}$ is the Euklidean norm.

? Can we do the above for functions? Idea $\sum_i \leftrightarrow \int_a^b$

Def: Let $a<b$: The <u>space of square integrable fct</u> is defined by

$$L^2([a,b]) = L^2(a,b) = \left\{ f:[a,b]\to\mathbb{R} \;:\; \int_a^b f(x)^2\,dx < \infty \right\}$$

<u>Inner product in L^2</u>: $\langle f,g\rangle_{L^2} = \int_a^b f(x)\cdot g(x)\,dx$

<u>Norm</u> $\|f\|_{L^2} = \sqrt{(f,f)_{L^2}} = \sqrt{\int_a^b f^2(x)\,dx}$

Rem: By def, $L^2(a,b) = \{ f:[a,b]\to\mathbb{R} : \|f\|_{L^2} < \infty \}$

Ex: Compute the L^2-norm of $f(x) = x^2$ on $[0,1]$:

(S) $\|f\|_{L^2} = \sqrt{\int_0^1 f(x)^2\,dx} = \sqrt{\int_0^1 x^4\,dx} = \sqrt{\frac{x^5}{5}\Big|_0^1} = \frac{1}{\sqrt{5}}$

As we did for vectors, one can define

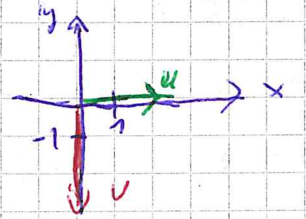
Def: Let $(V,(\cdot,\cdot)_V)$ be an inner product space

u and v are <u>orthogonal in V</u> if

$u \perp v \iff (u,v)_V = 0$

Ex: $V = \mathbb{R}^2$. Are the vectors $u = \binom{2}{0}$ and $v = \binom{0}{-3}$ orthogonal in V?

(S) $(u,v) = u^T v = (2,0)\binom{0}{-3} = 0+0 = 0 \rightsquigarrow$ yes!



Ex: $V = \mathcal{P}^{(2)}(-1,1)$. Are the "vectors" $p(x) = x$ and $q(x) = x^2$ orthog?

(S) $(p(x),q(x))_{L^2} \underset{\text{Def } L^2}{=} \int_{-1}^1 p(x)q(x)\,dx \underset{\text{Def } p,q}{=} \int_{-1}^1 x\cdot x^2\,dx = \int_{-1}^1 x^3\,dx = \frac{x^4}{4}\Big|_{-1}^1 = 0! \Rightarrow p\perp q.$

(P) **Th:** (<u>Cauchy-Schwarz inequality</u> (C-S))

Let $(V,(\cdot,\cdot))$ be an inner product space and $u,v\in V$. One has

$$|\langle u,v\rangle| \le \|u\|\cdot\|v\|$$

Proof: Let $\lambda\in\mathbb{R}$ and consider

$$0 \le \|u-\lambda v\|^2 \underset{\text{Def norm}}{=} (u-\lambda v, u-\lambda v) \underset{\text{linearity}}{=} (u,u) - \lambda \underset{(u,v)}{(v,u)} - \lambda(u,v) + \lambda^2 (v,v) =$$

$$= \|u\|^2 - 2\lambda(u,v) + \lambda^2\|v\|^2$$

This can be seen as a non-negativ polyn. of degree 2 in λ!
Hence " $b^2-4ac \leq 0$" or $(-2(u,v))^2 - 4\|u\|^2 \cdot \|v\|^2 \leq 0$ or

$4(u,v)^2 \leq 4\|u\|^2 \cdot \|v\|^2$ or $|(u,v)| \leq \|u\| \cdot \|v\|$:-) $\boxed{}$

Th: (Triangle inequality Δ)
Let $(V, (\cdot,\cdot))$ be an inner product space and $u,v \in V$. One has
$\|u+v\| \leq \|u\| + \|v\|$

Proof.

$$\underbrace{\|u+v\|^2 = (u+v, u+v)}_{\text{Def } \|\cdot\|} \underbrace{=}_{\text{linearity}} (u,u) + 2(u,v) + (v,v) \underbrace{=}_{\text{Def } \|\cdot\|} \|u\|^2 + 2(u,v) + \|v\|^2$$

$$\leq \|u\|^2 + 2|(u,v)| + \|v\|^2 \underbrace{\leq}_{C\text{-}S} \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 = (\|u\| + \|v\|)^2 \quad :-)$$

$\boxed{}$

4) Function spaces:

Since solutions to DE are fct, we introduce the following.

Let $a<b$.

Def: • The space of continuous fct on $[a,b]$ is denoted by

$$\mathcal{C}([a,b]) = \{ f : [a,b] \to \mathbb{R} : f \text{ is continuous} \}$$

• Similarly, one defines the space of (1) cont. diff. fct.

$$\mathcal{C}^{(1)}([a,b]) = \{ f : [a,b] \to \mathbb{R} : f \text{ and } f' \text{ are continuous} \}$$

$$\mathcal{C}^{(k)}([a,b]) = \{ f : [a,b] \to \mathbb{R} : f, f', .., f^{(k)} \text{ are continuous} \}$$

We equip the above spaces with the norms:

$$\|f\|_{\mathcal{C}([a,b])} = \max_{a \leq x \leq b} |f(x)| \qquad \forall f \in \mathcal{C}([a,b])$$

$$\|f\|_{\mathcal{C}^{(1)}([a,b])} = \|f\|_{\mathcal{C}([a,b])} + \|f'\|_{\mathcal{C}([a,b])} = \max_{a \leq x \leq b} \left(|f(x)| + |f'(x)| \right)$$

Rem: (Technicallity) see summary.

supremum of a set = least upper bnd of set

$(a,b) \Rightarrow$ replace max by sup

$\mathcal{C}^{(1)}([a,b]) = f$ cont on (a,b) s.t. its derivative can be cont. extended on $[a,b]$

Ex: Compute the $\mathcal{C}^{(0)}$ and $\mathcal{C}^{(1)}$ norms of $f(x) = x^2 + e^x$ for $x \in [0,2]$.

$$\|f\|_{\mathcal{C}} = \max_{0 \leq x \leq 2} |e^x + x^2| = 2^2 + e^2 = 4 + e^2 \,//$$

$\uparrow$ $0 \leq x^2 + e^x$ and increasing on $[0,2]$

$$\|f\|_{\mathcal{C}^1} = \max |e^x + x^2| + \max |e^x + 2x| = 4 + e^2 + 2 \cdot 2 + e^2 = 8 + 2e^2 \,//$$