13

$r \leftrightarrow$ source / sink [heater / window]

$u' \leftrightarrow$ convection / advection [transport of a chemical in fluid]

(P)

Let us finally consider another type of BC:

Solve the following BVP with cG(1) FEM:

$$(BVP)\quad \begin{cases} -a u''(x) + b u'(x) = r & \text{for } 0<x<1 \\ u(0)=0, \quad u'(1)=\beta \neq 0 \end{cases}$$

Here $a, b > 0$, $r, \beta \neq 0$ are given. WLOG $a=b=r=1$

Rem. Homogeneous Dirichlet BC in $x=0$

Inhomogeneous Neumann BC in $x=1$ ($u'(1)=\beta\neq 0$)

Prescribe FLUX at boundary

(i) To get a VF, consider the space $V = \{ v:[0,1]\to\mathbb{R} : v, v' \in L^2(0,1), v(0)=0 \}$ Dirichlet

Then, multiply DE by a test fct $v\in V$ and integrate:

$$-a\int_0^1 u''(x)v(x)\,dx + b\int_0^1 u'(x)v(x)\,dx = r\int_0^1 v(x)\,dx \overset{\text{by parts}}{\Leftrightarrow}$$

$$\underbrace{-a u'(x)v(x)\Big|_0^1}_{-a\, u'(1) v(1) + a\, u'(0) v(0)} + a\int_0^1 u'(x)v'(x)\,dx + b\int_0^1 u'(x)v(x)\,dx = r\int_0^1 v(x)\,dx$$

$u'(1) = \beta$ since Neumann, $v(0) = 0$ since $v\in V$

We get:

(VF) Find $u\in V$ s.t. $a(u', v')_{L^2} + b(u', v)_{L^2} = r\int_0^1 v(x)\,dx + a\beta v(1) \quad \forall v\in V$.

(ii) To get a FE problem consider

Uniform partition $\mathcal{T}_h$: $0 = x_0 < x_1 < \ldots < x_{m+1} = 1$ with $h = x_{j+1} - x_j = \frac{1}{m+1}$

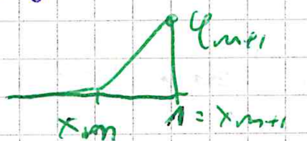
Space $V_h = \{ v:[0,1]\to\mathbb{R} : v \text{ is cont. pw. linear on } \mathcal{T}_h \text{ and } v(0)=0 \} = \text{span}(\varphi_1, \varphi_2, \ldots, \varphi_{m+1})$

We get

(FE) Find $u_h \in V_h$ s.t. $a\cdot(u_h', \chi')_{L^2} + b\cdot(u_h', \chi)_{L^2} = r\int_0^1 \chi(x)\,dx + a\beta\chi(1) \quad \forall \chi\in V_h$

(iii) To get the linear syst. of eq., choose $\chi = \varphi_i$, $i=1,2,\ldots,m+1$ and write $u_h(x) = \sum_{j=1}^{m+1} \xi_j \varphi_j(x)$ in (FE).

⚠ φ_{m+1} is half hat



The FE problem then reads:

$$a\cdot\Big(\sum_{j=1}^{m+1} \xi_j \varphi_j', \varphi_i'\Big)_{L^2} + b\cdot\Big(\sum_{j=1}^{m+1} \xi_j \varphi_j', \varphi_i\Big)_{L^2} = r\int_0^1 \varphi_i(x)\,dx + a\beta\varphi_i(1) \quad \text{for } i=1,2,\ldots,m+1.$$

$\Leftrightarrow$ (linearity)

$$a \sum_{j=1}^{m+1} \underbrace{(\varphi_j', \varphi_i')_{L^2}}_{S_{ij}} \cdot \xi_j + b \cdot \sum_{j=1}^{m+1} \underbrace{(\varphi_j', \varphi_i)_{L^2}}_{C_{ij}} \cdot \xi_j = \underbrace{r \int_0^1 \varphi_i(x)dx + a\beta\varphi_i(1)}_{r_i}$$

$\Leftrightarrow a \cdot S' \xi + b C' \cdot \xi = R \Leftrightarrow (aS' + bC')\xi = R$ (lin. syst. of eq.)

(5) $S' \leftrightarrow$ stiffness matrix $(m+1) \times (m+1)$

$$S' = \frac{1}{h}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & 2 & -1 \\ 0 & & -1 & \boxed{1} \end{pmatrix}$$

$\Delta\ \varphi_{m+1}$ half hat

$C' \leftrightarrow$ Convection matrix $(m+1) \times (m+1)$

$$C' = \frac{1}{2}\begin{pmatrix} 0 & 1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & 0 & 1 \\ 0 & & -1 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} rh \\ rh \\ \vdots \\ rh \\ \frac{rh}{2} + a\beta \end{pmatrix} \quad (m+1) \times 1$$

half hat

Neumann BC

Rem: Stiffness and mass matrices are symmetric :-)

Convection matrix is not :-(

3) Equivalence between (BVP) and (VF):

We consider the problem (f cont. + bounded)

$$(BVP) \begin{cases} -u''(x) = f(x) & \text{for } 0 < x < 1 & (DE) \\ u(0) = u(1) = 0 & & (BC) \end{cases}$$

and show that it is equivalent to

(VF) Find $u \in V^0$ s.t. $(u', v')_{L^2(0,1)} = (f, v)_{L^2(0,1)} \quad \forall v \in V^0$

if u is twice cont. diff. ($u \in C^2(0,1)$)

Recall: $V^0 = H_0^1(0,1) = \{v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1) \text{ and } v(0) = v(1) = 0\}$

Th.: (BVP) $\Leftrightarrow$ (VF) ~~[illegible]~~ and $u \in C^2(0,1)$

Proof:

$\Rightarrow$: Already done, multiply with test fct and integrate:

$-\int_0^1 u''(x)v(x)dx = \int_0^1 f(x)v(x)dx \overset{\text{by part}}{\Leftrightarrow} (u', v')_{L^2} = (f, v)_{L^2} \ \forall v \in V^0$

$\Leftarrow$: (VF) reads $(u',v')=(f,v)$ $\forall v\in U$ or after int. by part

$$-\int_0^1 u''(x)v(x)\,dx + \underbrace{u'(x)v(x)\Big|_0^1}_{=0 \text{ since } v\in V^0} = \int_0^1 f(x)v(x)\,dx \quad \text{or}$$

$$\int_0^1 (u''(x)+f(x))\,v(x)\,dx = 0 \quad \forall v\in U^0 \quad (*)$$

We show by contradiction that this implies (BVP) $u''(x)+f(x)=0$ on $(0,1)$.

Suppose $u''+f=0$, Then, by continuity, $\exists x\in(0,1)$ s.t. $u''+f\neq 0$ in a neighborhood of x. If we choose now $v\in V^0$ that has the same sign in this neighborhood and is zero everywhere else, we obtain $\int_0^1 (u''(x)+f(x))v(x)\,dx > 0$.

This is a contradiction to (*) !!

Therefore, we must have $u''(x)+f(x)=0$ on $(0,1)$, i.e. (BVP)

Def. We denote by $H_0^1(a,b)$ the space

$$H_0^1(a,b)=\{v:[a,b]\to\mathbb{R};\ v,v'\in L^2(a,b),\ v(a)=v(b)=0\}$$

4) Error estimates.

We start with

Th. (Poincaré inequality)

Let $L>0$ and the open interval $\Omega=(0,L)$. Assume $u\in H_0^1(\Omega)$, then

$$\|u\|_{L^2(\Omega)} \le C_L\cdot\|u'\|_{L^2(\Omega)}$$

Can bound u only with L and u'

Proof:

Since $u\in H_0^1(\Omega)$, then $u(0)=0$ and $u(x)=\int_0^x u'(y)\,dy$ $\forall x\in\Omega$.

It follows:

$$|u(x)|^2=\left|\int_0^x 1\cdot u'(y)\,dy\right|^2 \overset{C\text{-}S}{\le} \left|\int_0^x 1\,dy\right|\cdot\int_0^x |u'(y)|^2\,dy \overset{x\le L}{\le} x\cdot\int_0^L |u'(y)|^2\,dy$$

Finally, integrate wrt x and get

$$\underbrace{\int_0^L |u(x)|^2\,dx}_{\|u\|^2_{L^2(0,1)}} \le \underbrace{\int_0^L x\,dx}_{\frac{L^2}{2}}\cdot\int_0^L |u'(y)|^2\,dy \quad \text{or} \quad \|u\|^2_{L^2(0,1)} \le \underbrace{\frac{L^2}{2}}_{=\,C_L^2 \text{ (indep. of } u)}\cdot\|u'\|^2_{L^2(0,L)}$$

Def. For $f, g \in H_0^1(a,b)$, we define the energy inner product

$$(f,g)_E = \int_a^b f'(x)g'(x)\,dx = (f', g')_{L^2(a,b)}$$

and the energy norm

$$\|f\|_E = \sqrt{(f,f)_E} = \sqrt{(f',f')_{L^2(a,b)}} = \|f'\|_{L^2(a,b)}$$

The above definition is used to study error of FEM for

(BVP) $\begin{cases} -u''(x) = f(x) & \text{for } x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$

Recall,

(VF) Find $u \in H_0^1(0,1)$ s.t. $(u', v')_{L^2} = (f, v)_{L^2} \quad \forall v \in H_0^1(0,1)$

(FE) Find $u_h \in V_h^0$ s.t. $(u_h', \chi')_{L^2} = (f, \chi)_{L^2} \quad \forall \chi \in V_h^0$, where

Uniform partition T_h: $0 = x_0 < x_1 < \ldots < x_m < x_{m+1} = 1$, $x_{j+1} - x_j = h$

$V_h^0 = \text{span}(\varphi_1, \varphi_2, \ldots, \varphi_m)$, φ_j hat fct.

Obs $V_h^0 \subset H_0^1$ so that one can take $v = \chi$ in (VF)! And get

$$\left.\begin{array}{l} (u', \chi') = (f, \chi) \\ (u_h', \chi') = (f, \chi) \end{array}\right\} \Rightarrow (u' - u_h', \chi') = 0 \quad \forall \chi \in V_h^0 \; !$$

This is known as Galerkin orthogonality

Rem. The above means that the error $e = u - u_h$ of FEM is orthogonal to V_h^0 in the energy inner product.

This is now used to show

Th. Let u be the sol. to (BVP) and u_h the sol. to (FE). One has

$$\|u - u_h\|_E \leq \|u - \chi\|_E \quad \forall \chi \in V_h^0$$

That is the FE sol. is the best approx of u in V_h^0 in energy norm.

Proof.

Consider

$$\|u - u_h\|_E^2 \overset{\text{Def norm}}{=} (u' - u_h', u' - u_h')_{L^2} \overset{\chi \in V_h^0}{=} (u' - u_h', u' - \chi' + \chi' - u_h')_{L^2} \overset{\text{linearity}}{=}$$

$$= (u' - u_h', u' - \chi') + \underbrace{(u' - u_h', \underbrace{\chi' - u_h'}_{\in V_h^0})}_{=0 \text{ by Galerkin orthog.}} \overset{C\text{-}S}{\leq} \|u - u_h\|_E \cdot \|u - \chi\|_E \qquad (*)$$