Finally, we show

Th (a priori error estimate in energy norm)

Let u and u_h be sol. to (WF), resp. (FE). Assume $u \in C^2(0,1)$.

Then, $\exists\, C > 0$ s.t.

$$\|u - u_h\|_E \leq C \cdot h \cdot \|u''\|_{L^2}$$

Proof: above Th with $\chi = \Pi_h u \in V_h^0$ cont. pw linear interpolant of u

Start with $\|u - u_h\|_E \overset{\downarrow}{\leq} \|u - \Pi_h u\|_E = \|u' - (\Pi_h u)'\|_{L^2(0,1)} \leq$ (Def norm) (Chap. IV)

$$\leq C \cdot h \cdot \|u''\|_{L^2(0,1)}$$

[Rem: Not optimal $h \to h^2$!]

Chapter VI: Scalar initial value problems (IVP)

Goal: Study and approximate numerically solutions to

$$(\text{IVP}) \quad \begin{cases} \dot{y}(t) = f(y(t)) & 0 < t \leq T \\ y(0) = y_0, \end{cases}$$

where y_0, f, T given and $\dot{y}(t) = \frac{d}{dt} y(t)$.

1) Linear first order IVP

Consider the model problem

Population dynamic

$$(\text{IVP}) \quad \begin{cases} \dot{u}(t) + a(t)u(t) = f(t) \\ u(0) = u_0, \end{cases}$$

where f, u_0, a are given, f and a are continuous and $a(t) \geq 0$ bounded for instance.

Th: The sol. to (IVP) is given by the variation of constants formula (VOC)

$$u(t) = u_0 e^{-A(t)} + \int_0^t e^{-(A(t) - A(s))} f(s)\, ds,$$

where $A(t) = \int_0^t a(s)\, ds$ (easy if $a(t) = a$ constant)

Proof:

• Multiply (IVP) by integrating factor $e^{A(t)}$ to get

$$\underbrace{\dot u(t)e^{A(t)} + a(t)u(t)e^{A(t)}}_{\frac{d}{dt}\left(u(t)e^{A(t)}\right)} = f(t)e^{A(t)}$$

$$\frac{d}{dt}\left(u(t)e^{A(t)}\right) = \dot u(t)e^{A(t)} + u(t)e^{A(t)}\cdot \underbrace{\dot A(t)}_{=a(t)} \qquad \text{(prod. rule + chain rule)}$$

• Integrate above $\int_0^t \dots$:

$$u(t)e^{A(t)} - \underbrace{u(0)}_{=u_0}\cdot \underbrace{e^{A(0)}}_{e^0=1} = \int_0^t f(s)e^{A(s)}\,ds \Leftrightarrow u(t) = e^{-A(t)}u_0 + \int_0^t e^{-(A(t)-A(s))}f(s)\,ds$$

∎

We next show stability of solutions to (IVP):

used for error estimates

Th: Let $u(t)$ be the sol to (IVP). We have:

(i) If $a(t)\geq\alpha>0$, then $|u(t)|\leq e^{-\alpha t}|u_0| + \frac{1}{\alpha}(1-e^{-\alpha t})\max_{0\leq s\leq t}|f(s)|$

(dissipative case) ($t\to\infty$, no effect of $|u_0|$)

(ii) If $a(t)\geq 0$, then $|u(t)|\leq|u_0|+\int_0^t|f(s)|\,ds$

(parabolic case) effect of u_0, but bounded.

Proof:

(i) • Since $a(s)\geq\alpha>0$, one has $A(t) \overset{\text{Def } A}{=} \int_0^t a(s)\,ds \geq \int_0^t \alpha\, ds \geq \alpha t$.

Hence $A(t)\geq\alpha t$ and $-A(t)\leq-\alpha t$ or $e^{-A(t)}\leq e^{-\alpha t}$

Similarly, $A(t)-A(s)\geq\alpha(t-s)$ and $e^{-(A(t)-A(s))}\leq e^{-\alpha(t-s)}$ for $t>s$.

• Use the above in voc formula:

$$|u(t)| \overset{\triangle}{\leq} |u_0|e^{-A(t)} + \int_0^t e^{-(A(t)-A(s))}|f(s)|\,ds \leq |u_0|e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}\underbrace{|f(s)|}_{\leq \max_{0\leq\tilde s\leq t}|f(\tilde s)|}ds$$

$$\leq |u_0|e^{-\alpha t} + \max_{0\leq s\leq t}|f(s)| \underbrace{\int_0^t e^{-\alpha(t-s)}\,ds}_{\frac{1}{\alpha}e^{-\alpha(t-s)}\Big|_{s=0}^{t} = \frac{1}{\alpha}(1-e^{-\alpha t})}$$

(S)

$$\leq |u_0|e^{-\alpha t} + \max_{0\leq s\leq t}|f(s)|\cdot\frac{1}{\alpha}\cdot(1-e^{-\alpha t}) \qquad \text{ok!}$$

(ii) Since $a(t)\geq 0$, one has $A(t)=\int_0^t a(s)\,ds\geq\int_0^t 0\,ds\geq 0 \Rightarrow e^{-A(t)}\leq 1$

As above, one gets

$$|u(t)| \underset{VOC}{\overset{\Delta}{\leq}} |u_0| + \int_0^t |f(s)|\,ds \qquad :-)$$

Ex: Find the solution to following IVP:

(i) $\begin{cases} \dot{u}(t) + 2u(t) = t & 0 < t \leq 3 \\ u(0) = \frac{3}{4} \end{cases}$ (dissipative case)

(ii) $\begin{cases} \dot{u}(t) = t & 0 < t \leq 3 \\ u(0) = \frac{3}{4} \end{cases}$ (parabolic case)

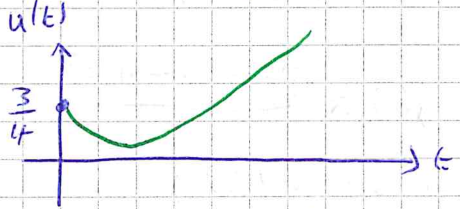
We use the VOC:

(i) $a(t) = 2 \Rightarrow A(t) = \int_0^t a(s)\,ds = \int_0^t 2\,ds = 2t$

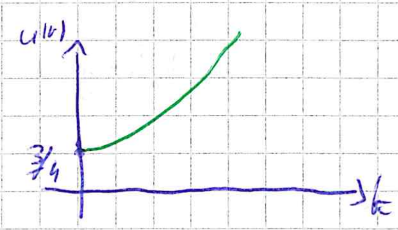
VOC gives

$$u(t) = \frac{3}{4}e^{-2t} + \int_0^t e^{-2(t-s)} s\,ds \overset{\text{by part}}{=} \dots = \frac{3}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}(1 - e^{-2t})$$

Hence $u(t) = e^{-2t} + \frac{t}{2} - \frac{1}{4}$



(ii) Direct $u(t) = \frac{t^2}{2} + \text{const} \underset{u(0)=\frac{3}{4}}{\Rightarrow} u(t) = \frac{t^2}{2} + \frac{3}{4}$



2) Finite differences (FD)

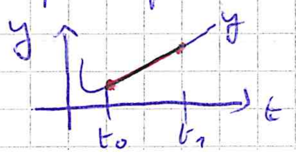
Consider $y: \mathbb{R} \to \mathbb{R}$ differentiable and let $t_0 \in \mathbb{R}$ fixed.

By def, one has $y'(t_0) = \lim_{h \to 0} \frac{y(t_0+h) - y(t_0)}{h}$. This motivates

Def: For a fixed $h > 0$, we define the forward difference by

$$y'(t_0) \approx \frac{y(t_0+h) - y(t_0)}{h} \quad \text{or} \quad y'(t_0) \approx \frac{y(t_1) - y(t_0)}{t_1 - t_0} \quad \text{for } t_1 = t_0 + h$$

Rem: The RHS is the slope of the line between $(t_0, y(t_0))$ and $(t_1, f(t_1))$



Similarly, one has the approximations

<u>Backward difference</u> $\quad y'(t_0) \approx \dfrac{y(t_0) - y(t_0 - h)}{h}$

<u>Centered difference</u> $\quad y'(t_0) \approx \dfrac{y(t_0+h) - y(t_0 - h)}{2h}$

3) First time integrators for IVPs

Consider

$$(\text{IVP}) \quad \begin{cases} \dot{y}(t) = f(y(t)) & \text{in } (0,T] \\ y(0) = y_0 \end{cases}$$

Let $N \in \mathbb{N}$ and define the <u>time step</u> $k = \frac{T}{N}$ as well as the grid

$0 = t_0 < t_1 < t_2 < \dots < t_N = T$, where $t_\ell = \ell \cdot k$ for $\ell = 0, 1, 2, \dots, N$

(P) <u>Idea</u>: Euler (1768)

For $t_0 \leq t \leq t_0 + h = t_1$, replace $\dot{y}(t)$ by forward difference

$$\underbrace{\dot{y}(t)}_{f(y(t))\ (\text{def IVP})} \approx \frac{y(t+k) - y(t)}{k} \quad \text{or} \quad y(t+k) \approx y(t) + k f(y(t))$$

Taking $t = t_0$, one gets $y(t_1) \approx y(t_0) + k \cdot f(y(t_0))$

$$\underset{\text{Def IVP}}{\approx} y_0 + k \cdot f(y_0) =: y_1$$

For the point $t_1 = t_0 + h$, we get approx. $y_1 = y_0 + k f(y_0) \approx y(t_1)$

We can repeat this procedure!!

$$\begin{cases} t_{\ell+1} = t_\ell + k \\ y_{\ell+1} = y_\ell + k f(y_\ell) \end{cases} \longrightarrow (\text{forward}) \ \underline{\text{Euler scheme}}!$$

<u>Rem</u>: This provides numerical approx $y_\ell \approx y(t_\ell)$ on grid $\{t_\ell\}$