Recall: ODE $\quad y'(x) + \sin(x) = 0$

PDE $\quad u_t(x,t) - u_{xx}(x,t) = 0$

Vector space / linear space

$\mathcal{P}^{(1)}(\mathbb{R}) = \{\text{polyn. of degree} \leq 1\} = \{a + bx : x \in \mathbb{R}\}$

Ex: (cont')

$\mathcal{P}^{(1)}(\mathbb{R})$ ok, similarly define $\mathcal{P}^{(n)}(\mathbb{R})$ to be the set of polynomials of degree $\leq n$.

For instance $\mathcal{P}^{(3)}(\mathbb{R})$ contains all polynomials of the form a_0, $a_0 + a_1 x$, $a_0 + a_1 x + a_2 x^2$, and $a_0 + a_1 x + a_2 x^2 + a_3 x^3$ $\forall a_0, a_1, a_2, a_3$ in $\mathbb{R}$.

Def: A subset $U \subset V$, where V is a vector space, is called a subspace if $\alpha u + \beta v \in U$ $\forall \alpha, \beta \in \mathbb{R}$ and $u, v \in U$.

Ex: • Let $V = \mathbb{R}^3$. Show that $U = \{(x, y, 0) : x, y \in \mathbb{R}\}$ is a subspace of V.

(i) $U \subset V$? clear!

(ii) $\alpha, \beta \in \mathbb{R}$, $u, v \in U \Rightarrow \alpha u + \beta v \in U$?

$\alpha u + \beta v = (\alpha x_1, \alpha y_1, 0) + (\beta x_2, \beta y_2, 0) =$

$= (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, 0) \in U$ Ok.

Ex: Let $V = \mathcal{P}^{(5)}(\mathbb{R})$ and $U = \mathcal{P}^{(1)}(\mathbb{R})$. Show that U is a subspace of V?

(i) $U \subset V$? Ok ($1 \leq 5$)

(ii) Let $\alpha, \beta \in \mathbb{R}$, $p(x), q(x) \in \mathcal{P}^{(1)}(\mathbb{R})$, $\alpha p(x) + \beta q(x) \in U$?

$\alpha p(x) + \beta q(x) = \alpha(a + bx) + \beta(c + dx) =$

$= (\alpha a + \beta c) + (\alpha b + \beta d)x \in U$ ok!

:-)

Def: Let V be a vector space and $v_1, v_2, \dots, v_n \in V$.

A linear combination of $v_1, v_2, \dots, v_n$ is given by $a_1 v_1 + a_2 v_2 + \dots + a_n v_n$ for some $a_1, a_2, \dots, a_n \in \mathbb{R}$

Def: Let V be a vector space and $v_1, v_2, \dots, v_n \in V$.

The space of all linear combinations

of $v_1, v_2, \dots, v_n$ is denoted by

$$\mathrm{Span}(v_1, v_2, \dots, v_n) = \{a_1 v_1 + a_2 v_2 + \dots + a_n v_n : a_1, \dots, a_n \in \mathbb{R}\}$$

Ex: $V = \mathbb{R}^2$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\mathrm{span}(v_1, v_2) = ?$

$$\mathrm{Span}(v_1, v_2) = \{ a_1 v_1 + a_2 v_2 : a_1, a_2 \in \mathbb{R} \} =$$

$$= \left\{ \begin{pmatrix} a_1 \\ 2a_2 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\} = \mathbb{R}^2$$

Ex: What is $\mathrm{span}\{1, x, x^2\}$?

degree 0 — polyn. of degree 1 — polyn. of degree 2

$$\mathrm{span}\{1, x, x^2\} \overset{\text{Def}}{=} \{ a_0 \cdot 1 + a_1 \cdot x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R} \}$$

$$= \mathcal{P}^{(2)}(\mathbb{R})$$

Def (since we generate all polyn. of degree ≤ 2)

Write the polynomial $4x - 2 \in \mathcal{P}^{(2)}(\mathbb{R})$

in terms of $1, x, x^2$:

$$4x - 2 = (-2) \cdot 1 + 4 \cdot x + 0 \cdot x^2$$

coordinates

2) Basis of vector spaces:

Let V be a vector space / linear space.

Def: A set $\{v_1, v_2, \dots, v_n\}$ in V is called linearly independent if the equation

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$$

has only the trivial solution $a_1 = a_2 = \dots = a_n = 0$.

Else it is called linearly dependent.

Ex: $V = P^{(2)}(\mathbb{R})$. Show that the polyn.

$p_1(x) = 1$, $p_2(x) = x$, $p_3(x) = x^2$ are lin. indep:

Consider equation: $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = 0$

where $a_1, a_2, a_3 \in \mathbb{R}$. This is just

$$a_1 \cdot 1 + a_2 \cdot x + a_3 x^2 = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$$

polynomial zero!!

The only possibility for this to happen is by taking $a_1 = a_2 = a_3 = 0$! $\rightsquigarrow$ lin. indep. :-)

Ex: Show that the polynomials $p_1(x) = 2 - x^2$, $p_2(x) = 3x$, $p_3(x) = x^2 + x - 2$ are lin. dependent in $\mathcal{P}^{(2)}(\mathbb{R})$.

Consider the equation: $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = 0$ (*)

$\Leftrightarrow a_1(2 - x^2) + a_2(3x) + a_3(x^2 + x - 2) = 0 \Leftrightarrow$

$\Leftrightarrow (2a_1 - 2a_3) \cdot 1 + (3a_2 + a_3)x + (a_3 - a_1)x^2 = 0$

$$\Leftrightarrow \begin{cases} 2a_1 - 2a_3 = 0 \\ 3a_2 + a_3 = 0 \\ a_3 - a_1 = 0 \end{cases} \Leftrightarrow \begin{matrix} a_1 = a_3 \\ a_3 = -3a_2 \end{matrix}$$

Taking, f. ex, $a_2 = 1 \rightsquigarrow a_3 = -3 \rightsquigarrow a_1 = -3$

we see that (*) is satisfied hence

p_1, p_2, p_3 are lin. dep. in $\mathcal{P}^{(2)}(\mathbb{R})$:-)

Rem: In the above example, one sees that one can express $p_2(x)$ in terms of $p_1(x)$ and $p_3(x)$: $p_2(x) = 3x = 3(p_3(x) + p_1(x))$!

This is always the case for lin. dep. objects.

<u>Def</u>: • A set $\{v_1, v_2, \dots, v_n\}$ in V is called a basis of V if

span$(v_1, v_2, \dots, v_n) = V$

set is linearly independent.

• The dimension of V, $\dim(V)$, is the number of elements in the basis ($\dim(V) = n$)

<u>Ex</u>: • The standard basis of $V = \mathbb{R}^2$ is

$e_1 = (1,0)$ and $e_2 = (0,1)$:

(i) $\mathrm{span}(e_1, e_2) = \{a_1 e_1 + a_2 e_2 : a_1, a_2 \in \mathbb{R}\}$

$= \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} : a_1, a_2 \in \mathbb{R} \right\} = \mathbb{R}^2$

(ii) e_1 and e_2 are lin. indep.:

Consider equation $a_1 e_1 + a_2 e_2 = 0$

$\Leftrightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow a_1 = a_2 = 0$!!

$\Rightarrow$ e_1 and e_2 are lin. indep. :-)

$\Rightarrow \dim(\mathbb{R}^2) = 2$ //

<u>Ex</u>: Show that $\{1, x, x^2\}$ is a basis for $\mathcal{P}^{(2)}(\mathbb{R})$.

(i) $\operatorname{span}\{1, x, x^2\} = \mathcal{P}^{(2)}(\mathbb{R})$? see above!

(ii) lin. indep. ? OK, see above!

(iii) $\dim(\mathcal{P}^{(2)}(\mathbb{R}))$ = number of element of $\{1, x, x^2\} = 3$

<u>Rem</u>: $\dim(\mathcal{P}^{(q)}(\mathbb{R})) = q + 1$

3) Scalar product / inner product:

Let V be a vect. sp. / lin. space.

Def: A scalar product or inner product on V is map $(\cdot,\cdot): V \times V \longrightarrow \mathbb{R}$ such that:

(i) $(u,v) = (v,u)$ (symmetry)

(ii) $(u+\alpha v, w) = (u,w) + \alpha(v,w)$ (linearity)

(iii) $(u,u) \geq 0$ (positivity)

(iv) $(u,u) = 0 \Leftrightarrow u = 0$

$\forall \alpha \in \mathbb{R}, \ \forall u,v,w \in V$

Def: $(V, (\cdot,\cdot))$ is called an inner product space.

- In such space, one defines

a norm: $\|u\| = \sqrt{(u,u)}$

Ex: If $V = \mathbb{R}^n$, $\underset{\overset{\|}{u \cdot v}}{(u,v)} = u^T v = \sum_{j=1}^{n} u_j v_j$

$\|u\| \rightsquigarrow$ Euclidean norm.

Def: The space of square integrable functions $f : [a,b] \to \mathbb{R}$ is

$$L^2([a,b]) = \left\{ f : [a,b] \to \mathbb{R} : \int_a^b |f(x)|^2 \, dx < \infty \right\}$$

Inner product: $(f,g)_{L^2} = \int_a^b f(x) \, (x) \, dx$

L^2-norm: $\|f\|_{L^2} = \sqrt{(f,f)_{L^2}} = \sqrt{\int_a^b f(x)^2 \, dx}$

$$\sqrt{\int_a^b f(x)^2 \, dx}$$