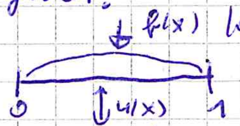
We give a detailed derivation of the FEM for the BVP (1)

$$(\text{BVP}) \quad \begin{cases} -(a(x)\,u'(x))' = f(x) & \text{for} \quad 0 < x < 1 \\ u(0) = 0, \; u(1) = 0, \end{cases}$$

where $a(x) > 0$ and $f(x)$ are (nice) given.

Simple model for load on bar



displacement bar

$a(x) \leftrightarrow$ modulus of elasticity

(i) In order to get a variational formulation (VF) of (BVP),

(S) we define $H_0^1 = \{ v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1), \; v(0) = v(1) = 0 \}$

homog. Dirichlet BC

Next: multiply (BVP) with test fct $v \in H_0^1$ and integrate over $[0,1]$:

$$-\int_0^1 (a(x)u'(x))'\,v(x)\,dx = \int_0^1 f(x)v(x)\,dx \overset{\text{by parts}}{\Longleftrightarrow}$$

$$\underbrace{-\big(a(x)\,u'(x)\,v(x)\big)\Big|_0^1}_{=0 \text{ since } v \in H_0^1} + \int_0^1 a(x)u'(x)\,v'(x)\,dx = \int_0^1 f(x)\,v(x)\,dx \quad \forall v \in H_0^1$$

Finally, the VF reads:

$$(\text{VF}) \quad \text{Find } \underset{\text{trial space}}{u \in H_0^1} \text{ s.t. } \int_0^1 a(x)u'(x)v'(x)\,dx = \int_0^1 f(x)\,v(x)\,dx \quad \forall v \in \underset{\text{test space}}{H_0^1}$$

Rem. Here, trial = test. Not always the case, see below.

(ii) To get a FE problem, we consider the finite dimensional space

$V_h^0 = \{ v : [0,1] \to \mathbb{R} : v \text{ is cont and pw linear on } \mathcal{T}_h, \; v(0) = v(1) = 0 \} \subset H_0^1$,

where $\mathcal{T}_h$ uniform partition: $0 = x_0 < x_1 < \ldots < x_{m+1} = 1$ and $h = \frac{x_{m+1} - x_0}{m+1} = \frac{1}{m+1}$.

(S) Recall $V_h^0 = \text{span}(\varphi_1, \varphi_2, \ldots, \varphi_m)$ with hat fct φ_j.

Finally, the FE problem reads:

Notation $\chi \leftrightarrow v$ better?

$$(\text{FE}) \quad \text{Find } u_h \in V_h^0 \text{ s.t. } \int_0^1 a(x)\,u_h'(x)\,\chi'(x)\,dx = \int_0^1 f(x)\,\chi(x)\,dx \quad \forall \chi \in V_h^0$$

Rem. This is called a cG(1) FE for continuous Galerkin (linear).

(iii) To write (FE) as a linear syst. of eq., take $\chi = \varphi_i$ for $i = 1, 2, \ldots, m$ and write $u_h(x) = \sum_{j=1}^m \xi_j \varphi_j(x)$. The FE prob. reads

Find $\zeta_1, \zeta_2, \dots, \zeta_m$ s.t. $\int_0^1 a(x)\left(\sum_{j=1}^m \zeta_j \varphi_j'(x)\right)\varphi_i'(x)\,dx = \int_0^1 f(x)\varphi_i(x)\,dx$ for $i=1,\dots,m$

$$\Leftrightarrow \sum_{j=1}^m \underbrace{\left(\int_0^1 a(x)\varphi_j'(x)\varphi_i'(x)\,dx\right)}_{s_{ij}} \zeta_j = \underbrace{\int_0^1 f(x)\varphi_i(x)\,dx}_{b_i} \quad \text{for } i=1,2,\dots,m.$$

This gives a syst. of eq. $S\zeta = b$, where

$S = (s_{ij})_{i,j=1}^m$ ⇝ stiffness matrix

$b = (b_i)_{i=1}^m$ ⇝ load vector

$\zeta = (\zeta_j)_{j=1}^m$ ⟶ unknown vector

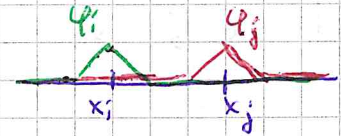
One can compute the entries of S and get (details ⇝ book)

$$s_{i,i} = \frac{1}{h^2}\int_{x_{i-1}}^{x_i} a(x)\,dx + \frac{1}{h^2}\int_{x_i}^{x_{i+1}} a(x)\,dx$$

OBS: If $a(x)$ complicated ⇝ quadrature rule!!

$$s_{i,i+1} = s_{i+1,i} = -\frac{1}{h^2}\int_{x_i}^{x_{i+1}} a(x)\,dx \quad \text{for } i=1,2,\dots,m-1$$

$$s_{ij} = 0 \quad \text{for } |i-j|>1$$



$$\hookrightarrow S = \begin{pmatrix} * & \times & & & 0 \\ \times & * & \times & & \\ & \times & * & \times & \\ & & \times & * & \times \\ 0 & & & \times & * \end{pmatrix}$$

Similarly, the entries of the load vector are

$$b_i = \int_{x_{i-1}}^{x_i} f(x)\varphi_i(x)\,dx + \int_{x_i}^{x_{i+1}} f(x)\varphi_i(x)\,dx$$

OBS: quad. rule if needed!

Solving the linear syst. $S\zeta = b$ gives ζ and then the FE sol. $u_h(x)$!

2) FEM for convection-diffusion-absorption BVP:

Goal: Adapt previous section to more general BVP or BC.

For instance, consider

$$(BVP)\begin{cases} -u''(x) + 4u(x) = 0 & \text{for } 0<x<1 \\ u(0) = \alpha,\ u(1) = \beta, \end{cases}$$

$u'' \leftrightarrow$ diffusion (Heat in Room)

$u \leftrightarrow$ absorption

where $\alpha, \beta \neq 0$ are given.

P **Obs:** The above BC are called non-homogeneous Dirichlet BC

(i) For the VF, consider

trial space: $V = \{ v : [0,1] \to \mathbb{R} \;:\; v, v' \in L^2(0,1),\; v(0) = \alpha,\; v(1) = \beta \}$ (because of BC),

test space: $V^0 = \{ v : [0,1] \to \mathbb{R} \;:\; v, v' \in L^2(0,1),\; v(0) = 0,\; v(1) = 0 \}$

They are different!

S Then, multiply (BVP) with $v \in V^0$ and integrate by part to get:

$$-\int_0^1 u''(x) v(x)\,dx + 4\int_0^1 u(x)v(x)\,dx = 0 \overset{\text{by part}}{\iff} \underbrace{-\left(u'(x)\,v(x)\right)\Big|_0^1}_{-(u'(1)v(1) - u'(0)v(0)) = 0 \text{ for } v \in V^0!} + \int_0^1 u'(x)v'(x)\,dx + 4\int_0^1 u(x)v(x)\,dx = 0$$

Finally, the VF reads

S (VF) Find $u \in V$ s.t. $\displaystyle\int_0^1 u'(x)v'(x)\,dx + 4\int_0^1 u(x)v(x)\,dx = 0 \quad \forall v \in V^0$

(ii) To get a FE problem, we consider a uniform partition $\mathcal{T}_h$ with $h = \frac{1}{m+1}$ and the FE spaces, where φ_j are hat fct,

$$V_h = \{ v : [0,1] \to \mathbb{R} : v \text{ cont. pw linear on } \mathcal{T}_h \text{ and } v(0) = \alpha,\; v(1) = \beta \} = \\ = \operatorname{span}(\varphi_0, \varphi_1, \ldots, \varphi_{m+1})$$

$$V_h^0 = \{ v : [0,1] \to \mathbb{R} : v \text{ cont. pw linear on } \mathcal{T}_h \text{ and } v(0) = 0 = v(1) \} = \\ = \operatorname{span}(\varphi_1, \varphi_2, \ldots, \varphi_m)$$

no need of φ_1 and φ_{m+1}!

S **Rem:** $V_h \subset V$ and $V_h^0 \subset V^0$ and $\dim(V_h) = m+2$, $\dim(V_h^0) = m$

Finally, the FE problem reads

(FE) Find $u_h \in V_h$ s.t. $\displaystyle\int_0^1 u_h'(x)\chi'(x)\,dx + 4\int_0^1 u_h(x)\chi(x)\,dx = 0 \quad \forall \chi \in V_h^0.$

(iii) To write (FE) as a linear syst. of eq., we set

$$u_h(x) = \sum_{j=0}^{m+1} \zeta_j \varphi_j(x) \qquad \text{since } u_h \in V_h = \operatorname{span}(\varphi_0, \varphi_1, \ldots, \varphi_{m+1}).$$

S Furthermore, $u_h(0) = u_h(x_0) \overset{!}{=} \alpha = \zeta_0 \underbrace{\varphi_0(x_0)}_{1} + \zeta_1 \underbrace{\varphi_1(x_0)}_{0} + \zeta_2 \underbrace{\varphi_2(x_0)}_{0} + \ldots + \zeta_{m+1} \underbrace{\varphi_{m+1}(x_0)}_{0}$

$= \zeta_0$ hence $\zeta_0 = \alpha$!

Similarly $u_h(1) \overset{!}{=} \beta$ implies $\zeta_{m+1} = \beta$.

Thus, one has $u_h(x) = \alpha\varphi_0(x) + \sum_{j=1}^{m} \underbrace{\zeta_j}_{\text{unknowns!!}} \varphi_j(x) + \beta\varphi_{m+1}(x)$

We insert the above into (FE) and take test fct $\chi = \varphi_i$, $i = 1, \ldots, m$;

$$\int_0^1 \Big(\alpha\varphi_0'(x) + \sum_{j=1}^{m} \zeta_j \varphi_j'(x) + \beta\varphi_{m+1}'(x)\Big)\varphi_i'(x)\,dx + 4\int_0^1 \Big(\alpha\varphi_0(x) + \sum_{j=1}^{m} \zeta_j \varphi_j(x) + \beta\varphi_{m+1}(x)\Big)\varphi_i(x)\,dx = 0$$

$\Leftrightarrow$

$$\sum_{j=1}^{m} \zeta_j \Big(\underbrace{\int_0^1 \varphi_j'(x)\varphi_i'(x)\,dx}_{s_{ij}}\Big) + 4\sum_{j=1}^{m} \zeta_j \Big(\underbrace{\int_0^1 \varphi_j(x)\varphi_i(x)\,dx}_{m_{ij}}\Big) + \alpha\int_0^1 \varphi_0'(x)\varphi_i'(x)\,dx + \beta\int_0^1 \varphi_{m+1}'(x)\varphi_i'(x)\,dx +$$
$$+ 4\alpha\int_0^1 \varphi_0(x)\varphi_i(x)\,dx + 4\beta\int_0^1 \varphi_{m+1}(x)\varphi_i(x) = 0$$

$\Leftrightarrow$

$$\sum_{j=1}^{m} \zeta_j s_{ij} + 4\sum_{j=1}^{m} \zeta_j m_{ij} = \underbrace{-\alpha(\varphi_0', \varphi_i')_{L^2} - \beta(\varphi_{m+1}', \varphi_i')_{L^2} - 4\alpha(\varphi_0, \varphi_i)_{L^2} - 4\beta(\varphi_{m+1}, \varphi_i)}_{b_i}$$

Finally, we obtain the linear syst. of eq

$$S\zeta + 4M\zeta = b \quad \text{or} \quad (S + 4M)\zeta = b, \text{ where}$$

$$S = \frac{1}{h}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix} \quad m\times m \text{ stiffness matrix}$$

$$M = \frac{h}{6}\begin{pmatrix} 4 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 4 \end{pmatrix} \quad m\times m \text{ mass matrix (Book + Lab)}$$

$$b = \left(\frac{1}{h} - \frac{2h}{3}\right)\begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \\ \beta \end{pmatrix} \quad m\times 1 \text{ load vector}$$

$\zeta \rightsquigarrow m\times 1$ vector of unknowns!

Solving for ζ gives us the FE sol u_h :-)