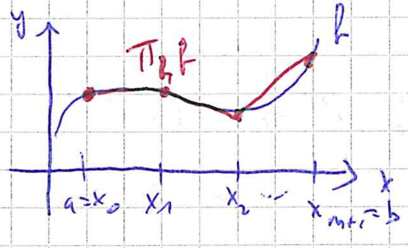
$$\Pi_h f(x) = \sum_{j=0}^{m+1} f(x_j)\,\varphi_j(x) \quad \text{for } x\in[a,b]$$

Illustration:

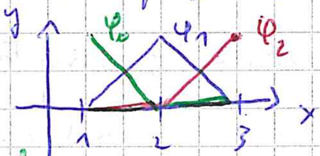


Ex. Let $f(x)=(x-1)(x-2)+1$ for $x\in[1,3]$ and uniform part. of $[1,3]$ into 2 subintervals. Find $\Pi_h f$.

S One has $\tau_h: 1=x_0 < \underset{2}{x_1} < x_2 = 3$ and thus $m=1$

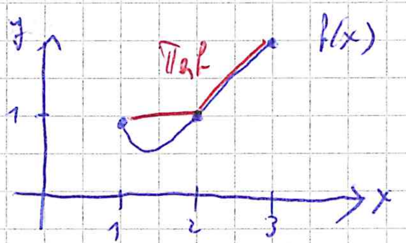
Furthermore, $f(x_0)=1$, $f(x_1)=1$ and $f(x_2)=3$.

We use the hat fct.:



We thus obtain $\Pi_h f(x) \overset{Def}{=} f(x_0)\varphi_0(x) + f(x_1)\varphi_1(x) + f(x_2)\varphi_2(x) =$

$$= \varphi_0(x)+\varphi_1(x)+3\varphi_2(x) = \begin{cases} \frac{x-x_1}{-h} + \frac{x-x_0}{h} + 0 = 1 & \text{for } x_0 \le x < x_1 \\ 0 + \frac{x-x_2}{-h} + 3\,\frac{x-x_1}{h} = 2x-3 & \text{for } x_1 \le x \le x_2 \end{cases}$$



⚠ h smaller / $m \uparrow$ ⇝ better approximation

What about the error of such interpolation?

Th: Let $f\in C^2(a,b)$ and $\Pi_h f$ as above. Then, $\exists$ constants C_1, C_2, C_3 s.t.

1) $\|\Pi_h f - f\|_{L^p(a,b)} \le C_1 \cdot \boxed{h^2}\, \|f''\|_{L^p(a,b)}$

2) $\|\Pi_h f - f\|_{L^p(a,b)} \le C_2\, h\, \|f'\|_{L^p(a,b)}$ for $p=1,2,\infty$

3) $\|(\Pi_h f)' - f'\|_{L^p(a,b)} \le C_3 \cdot h\, \|f''\|_{L^p(a,b)}$

Proof (of 1) for $p=1,2$)

S $$\|\Pi_h f - f\|^p_{L^p(a,b)} \underset{\text{Def norm}}{=} \int_a^b |\Pi_h f(x) - f(x)|^p\,dx \underset{\uparrow}{=} \sum_{j=1}^{m+1} \int_{x_{j-1}}^{x_j} |\Pi_h f(x) - f(x)|^p\,dx =$$

$a=x_0<x_1<\dots<x_{m+1}=b$

$$= \sum_{j=1}^{m+1} \|\Pi_h f - f\|^p_{L^p(x_{j-1},x_j)} \overset{\text{prev. Th.}}{\leq} \sum_{j=1}^{m+1} C_1^p (h^2)^p \|f''\|^p_{L^p(x_{j-1},x_j)} \leq$$

Def norm   linear interpol. from previous section!

$$\leq (C_1 h^2)^p \sum_{j=1}^{m+1} \int_{x_{j-1}}^{x_j} |f''(x)|^p \, dx = (C_1 h^2)^p \|f''\|^p_{L^p(a,b)}$$

Def norm   Def τ_h

Finally, take $1/p$ $\rightarrow$ $\|\Pi_h f - f\|_{L^p(a,b)} \leq C_1 \cdot h^2 \|f''\|_{L^p(a,b)}$;-)

<u>Ex:</u> (see previous expl)

Consider $f(x) = (x-1)(x-2)+1 = x^2-3x+3$ on $[1,3]$ and $\Pi_h f \in V_h(1,3)$

Find a bound for the error $\|\Pi_h f - f\|_{L^p(1,3)}$ for $p = 1, 2, \infty$.

Start by computing $f''(x) = 2$ and, ~~using~~ see above theorem,

$\|f''\|_{L^1(1,3)} = \int_1^3 |f''(x)| dx = 2 \cdot 2 = 4.$

$\|f''\|_{L^2(1,3)} = 2\sqrt{2}$, $\|f''\|_{L^\infty(1,3)} = \dots = 2$

This gives us $\|\Pi_h f - f\|_{L^p(1,3)} \leq C \cdot h^2 \cdot \|f''\|_{L^p(1,3)} = \begin{cases} 4C & \text{for } p=1 \\ 2\sqrt{2}C & \text{" } p=2 \\ 2C & \text{" } p=\infty \end{cases}$

We took $h=1$ in expl.

3) Lagrange interpolation:

<u>Recall:</u> Consider $q+1$ distinct points $a = x_0 < x_1 < \dots < x_q = b$ and Lagrange polyn. $\lambda_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{q} \frac{x-x_j}{x_i - x_j}$ for $i = 0, 1, \dots, q$

<u>Obs</u> $\lambda_j(x_i) = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$

<u>Obs</u> $\mathcal{P}^{(q)}(a,b) = \text{span}(\lambda_0, \lambda_1, \dots, \lambda_q)$, hence

$$p(x) = \sum_{j=0}^{q} p(x_j)\lambda_j(x) \quad \forall p \in \mathcal{P}^{(q)}(a,b).$$

<u>Def.</u> For $f : [a,b] \rightarrow \mathbb{R}$ continuous, one defines Lagrange interpolant

$$\Pi_q f(x) = \sum_{j=0}^{q} f(x_j)\lambda_j(x)$$

<u>Ex:</u> Consider $f(x) = x^3 + 1$ for $x \in [0,2]$. Lagrange polyn $\Pi_1 f$ and $\Pi_2 f$?

3 (i) For the linear interpolant, we have $x_0 = 0$, $x_1 = 2$

and thus $\lambda_0(x) = \dfrac{x - x_1}{x_0 - x_1} = \dfrac{x-2}{-2} = 1 - \dfrac{x}{2}$ and $\lambda_1(x) = \dfrac{x - x_0}{x_1 - x_0} = \dfrac{x}{2}$

Hence, $\Pi_1 f(x) \underset{\text{Def } \Pi_1 f}{=} f(x_0)\lambda_0(x) + f(x_1)\lambda_1(x) = \ldots = 4x + 1$ //

(ii) Similarly, for the quadratic interpolant $\Pi_2 f$, one uses

$x_0 = 0$, $x_1 = 1$, $x_2 = 2$ and $\lambda_0(x) = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{1}{2}(x-1)(x-2)$,

$\lambda_1(x) = -x(x-2)$, $\lambda_2(x) = \frac{1}{2}x(x-1)$ to get

S $\Pi_2 f(x) = \underbrace{f(x_0)}_{1} \cdot \lambda_0(x) + \underbrace{f(x_1)}_{2} \cdot \lambda_1(x) + \underbrace{f(x_2)}_{9} \cdot \lambda_2(x) = \ldots = 3x^2 - 2x + 1$ //

4) Numerical integration / quadrature rules:

Prob. Approximate integral $\int_a^b f(x)\,dx$.

Application: Load vector in FE discretisation of BVP

Idea, Recall: polyn. interpolation approximate fct: $\Pi_q f \approx f$

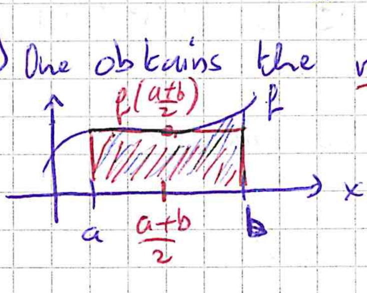
Hence $\int_a^b f(x)\,dx \approx \int_a^b \underbrace{\Pi_q f(x)}_{\text{easy to integrate (polyn.)}}\,dx$

The following are classical expl. of quadrature rules

(i) $q = 0$: $f(x) \approx f\left(\frac{a+b}{2}\right)$ [constant polyn], this gives

$$\int_a^b f(x)\,dx \approx \int_a^b f\left(\tfrac{a+b}{2}\right)dx \approx (b-a)\, f\left(\tfrac{a+b}{2}\right)$$

↳ One obtains the midpoint rule $\int_a^b f(x)\,dx \approx (b-a)\, f\left(\frac{a+b}{2}\right)$



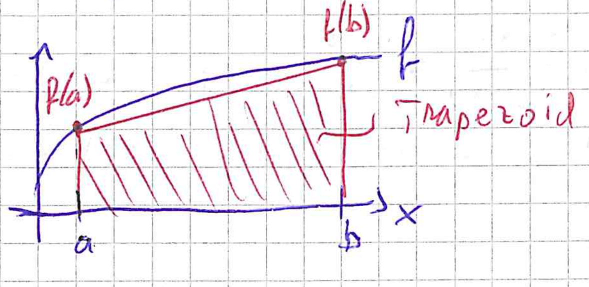
(ii) $q = 1$, $x_0 = a$, $x_1 = b$ and Lagrange interpolant $\Pi_1 f(x) = f(x_0)\lambda_0(x) + f(x_1)\lambda_1(x)$

$= f(a)\dfrac{x-b}{a-b} + f(b)\dfrac{x-a}{b-a}$.

Thus $\int_a^b f(x)\,dx \approx \int_a^b \Pi_1 f(x)\,dx = \dfrac{f(a)}{a-b}\int_a^b (x-b)\,dx + \dfrac{f(b)}{(b-a)}\int_a^b (x-a)\,dx = \dfrac{b-a}{2}\left(f(a) + f(b)\right)$,

Better if
f periodic
f rough

This gives the trapezoidal rule $\int_a^b f(x)dx \approx \frac{b-a}{2}\left(f(a)+f(b)\right)$



(iii) Similarly, Simpson's rule reads $\int_a^b f(x)dx \approx \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$

[details in book]

Ex: Approximate $\int_0^{\pi/4} \sin(x)\,dx$ by midpoint, trapez, , Simpson.

Midpoint: $\int_0^{\pi/4} \underbrace{\sin(x)}_{f(x)}dx \approx \left(\frac{\pi}{4}-0\right)\cdot f\left(\frac{\pi}{8}\right) = \frac{\pi}{4}\sin\left(\frac{\pi}{8}\right) \approx 0.3$

Trapez: $\int_0^{\pi/4} \sin(x)dx \approx \frac{\pi/4-0}{2}\left(\sin(0)+\sin(\pi/4)\right) \approx 0.27$

Simpson: $\int_0^{\pi/4} \sin(x)dx \approx \frac{\pi/4-0}{6}\left(\sin(0)+4\sin\left(\frac{\pi}{8}\right)+\sin\left(\frac{\pi}{4}\right)\right) \approx 0,2929$

Obs, Exact value $0,29289$

⚠

In practice, one apply a quadrature rule on a small interval, hence one consider first a partition:

$a = x_0 < x_1 < \dots < x_N = b$ with $x_{j+1}-x_j = h_j$ small

and then look at:

$$\int_a^b f(x)dx = \sum_{j=0}^{N-1} \underbrace{\int_{x_j}^{x_{j+1}} f(x)dx}_{\text{apply quadrature rule HERE!!}} \overset{\text{midpoint}}{\approx} \sum_{j=0}^{N-1} f\left(\frac{x_{j+1}+x_j}{2}\right)(x_{j+1}-x_j)$$

Chapter V: FEM for two-point BVP

Goal: Use all the above tools and spaces to present and analyse FEM for BVP.

1) Finite element method: