Chapter VIII : Laplace transforms

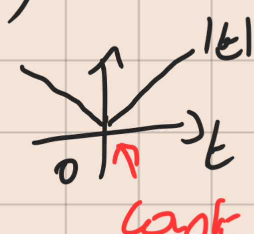
Goal: Present and study a tool to find exact solutions to particular DE.

1) Preliminaries:

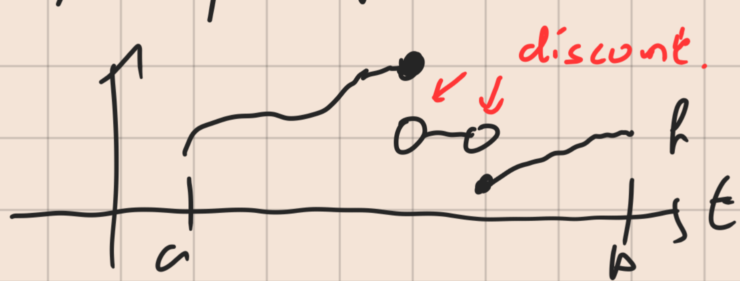
Def: A function $f : [a,b] \to \mathbb{R}$ is called piecewise continuous if it has only a finite number of discontinuities and at each of these discontinuity points the limits from left and from right $\exists$.

Ex: • f continuous $\Rightarrow$ f pwc (= piecewise cont.)

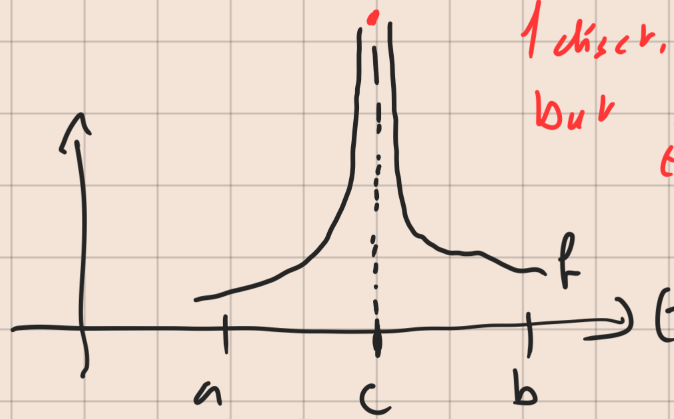
Ex: Hat functions, Lagrange fct, polyn., , $|t|$



• Graph of a pwc fct looks like



• Graph of a fct that is not pwc looks like



Def: A fct $f:[0,\infty) \to \mathbb{R}$ is of exponential order if $\exists$ constants $M, a, T > 0$ s.t.

$$|f(t)| \leq M \cdot e^{at} \quad \forall\, t \geq T.$$

Ex: • Graph

• All nice functions: $\sin(t)$, $\cos(22t)$, t^{55}, e^{27t}, ...

Ex: $f(t) = \sin(t)$, we know $|\sin(t)| \leq 1$

Here, one can take $M = 1$ and $a = 0$ to get $M \cdot e^{at} = 1 \cdot 1 = 1$

- A fct that is not of exponential order

is $f(t) = e^{t^2}$ ("t^2" grow faster that "at")

$\exp(t^2)$ $\exp(at)$

Def: A function f is called causal if

$f(t) = 0$ for $t < 0$.

Ex: (Heaviside function)

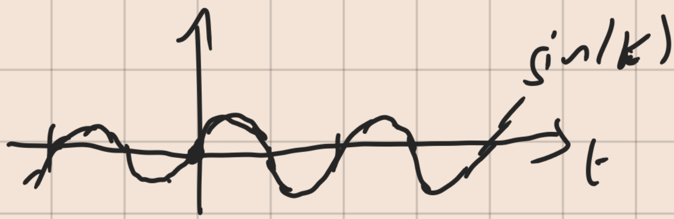
The Heaviside fct / step fct is defined as

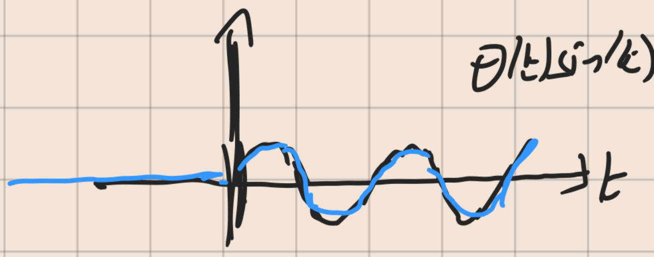
$$\theta(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Rem: If f is not causal, one can just

consider $f(t) \cdot \theta(t)$;

Ex: Look at $f(t) = \sin(t)$





Ex/Def: For a function $f(t)$, one can shift it to the left $f(t+T)$

the right $f(t-T)$

$\theta(t-2)$

2) Laplace transform:

Def: The Laplace transform (LT) of $f: [0,\infty) \to \mathbb{R}$ is defined by

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

(defined when integral $\exists$)

Ex: • Compute LT of $\theta(t)$:

$$\mathcal{L}\{\theta\}(s) \overset{\text{Def LT}}{=} \int_0^\infty \theta(t) e^{-st}\, dt \overset{\text{Def } \theta(t)}{=} \int_0^\infty 1 \cdot e^{-st}\, dt =$$

$$= \lim_{T \to \infty} \frac{e^{-st}}{-s} \Bigg|_{t=0}^{T} =$$

$$= \lim_{T \to \infty} \left(\frac{e^{-sT}}{-s} - \frac{1}{-s} \right) =$$

$$= -\frac{1}{s} \lim_{T \to \infty} \left(e^{-sT} \right) + \frac{1}{s} =$$

$$= 0 + \frac{1}{s} = \frac{1}{s} \text{ for } \underline{s > 0}$$

for $s > 0 \Rightarrow$ "neg. exp." and limit $= 0$

for $s < 0 \Rightarrow$ limit $\nexists$

$\Rightarrow$ LT of $\theta(t)$ reads $\mathcal{L}\{\theta\}(s) = \frac{1}{s}$ for $s > 0$.

Ex: Compute the LT of $f(t) = e^{ct}$, where $c \in \mathbb{R}$.

$$\mathcal{L}\{f\}(s) \underset{\text{Def LT}}{=} \int_0^\infty f(t)\, e^{-st}\, dt \underset{\text{Def } f}{=} \int_0^\infty e^{(c-s)t}\, dt =$$

$$= \lim_{T \to \infty} \left(\frac{e^{(c-s)t}}{c-s} \Bigg|_{t=0}^{T} \right) =$$

$$= \lim_{T \to \infty} \left(\frac{e^{(c-s)T}}{c-s} \right) - \frac{1}{c-s} \underset{\substack{\text{only } \exists \\ \text{if } c-s<0 \\ \text{or } c<s}}{=} 0 - \frac{1}{c-s}$$

$$= \frac{1}{s-c} \quad \text{for} \quad s > c.$$

Th: Assume that $f(t)$, $t \geq 0$, is pwc and of exponential order. Then, the LT $\mathcal{L}\{f\}(s)$ $\exists$.

Proof:

$$\mathcal{L}\{f\}(s) \underset{\text{Def LT}}{\overset{f \text{ pwc}}{=}} \int_0^\infty f(t) e^{-st}\, dt \overset{x \leq |x|}{\leq} \int_0^\infty |f(t)| e^{-st}\, dt$$

$$\underset{\text{Def of exp. order}}{\leq} \int_{0\,(T)}^\infty M e^{at} e^{-st}\, dt \leq M \int_{0\,(T)}^\infty e^{(a-s)t}\, dt$$

Def of exp. order !

$$\underset{\text{previous example}}{\leq} M \cdot \frac{1}{s-a} < \infty \text{ for } s > a$$

∎

Properties of Laplace transform:

Th: LT is a linear operator:

$$\mathcal{L}\{a\cdot f + b\cdot g\}(s) = a\,\mathcal{L}\{f\}(s) + b\cdot\mathcal{L}\{g\}(s)$$

$\forall a, b \in \mathbb{R}$, $\forall f, g$ fct

(when all LT $\exists$)

Proof: Clear since LT are integrals and integrals are linear. $\blacksquare$

Ex: • Compute $\mathcal{L}\{4\cdot\theta(t) + 3e^{-2t}\}(s) = ?$

• One has

$$\mathcal{L}\{4\theta(t) + 3e^{-2t}\}(s) \overset{\text{linearity}}{=}$$

$$= 4\cdot\mathcal{L}\{\theta(t)\}(s) + 3\mathcal{L}\{e^{-2t}\}(s) =$$

exempel above / Table

$\overset{b}{=} 4 \cdot \frac{1}{s} + 3 \cdot \frac{1}{s-(-2)} =$

$(s>0)$ $\quad$ $(s>-2)$

$$= \frac{4}{s} + \frac{3}{s+2} \quad \text{for } s>0$$

Ex: • Compute LT of $\sin(at)$ and $\cos(at)$, where $a \in \mathbb{R}$.

• Idea: Euler formula:

$$e^{iat} = \cos(at) + i\sin(at)$$

$$i = \sqrt{-1}$$

Take LT above:

$$\mathcal{L}\{e^{iat}\}(s) \underset{\substack{4 \\ \text{linearity}}}{=} \mathcal{L}\{\cos(at)\}(s) + i\,\mathcal{L}\{\sin(at)\}(s)$$

Using previous expl./Table,

$$\underbrace{\frac{1}{s-ia}} = \mathcal{L}\{\cos(at)\}(s) + i\,\mathcal{L}\{\sin(at)\}(s)$$

$$\frac{1}{s-ia}\underbrace{\frac{s+ia}{s+ia}}_{=1} = \frac{s+ia}{s^2+a^2}$$

This gives

$$\underline{\frac{s}{s^2+a^2}} + i\,\underline{\frac{a}{s^2+a^2}} = \underline{\mathcal{L}\{\cos(at)\}(s)} + i\,\underline{\mathcal{L}\{\sin(at)\}(s)}$$

$$\Rightarrow \mathcal{L}\{\cos(at)\}(s) = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{\sin(at)\}(s) = \frac{a}{s^2+a^2}$$

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