Recall:

$$(IVP)\quad \begin{cases} \dot{y}(t) = f(y(t)) \\ y(0) = y_0 \end{cases}, \quad 0 < t \leq T$$

$f: \mathbb{R} \to \mathbb{R}$, y_0 given

$\dot{y}(t) = \frac{d}{dt} y(t)$

Taylor: $y(t+k) = y(t) + \dot{y}(t)k + \dots = y(t) + k f(y(t)) + \dots$

$$\Rightarrow y(t+k) \approx y(t) + k f(y(t))$$

For $t = t_0$: $y(t_1) \approx y_0 + k \cdot f(y_0) =: y_1 \Rightarrow y_1 \approx y(t_1)$

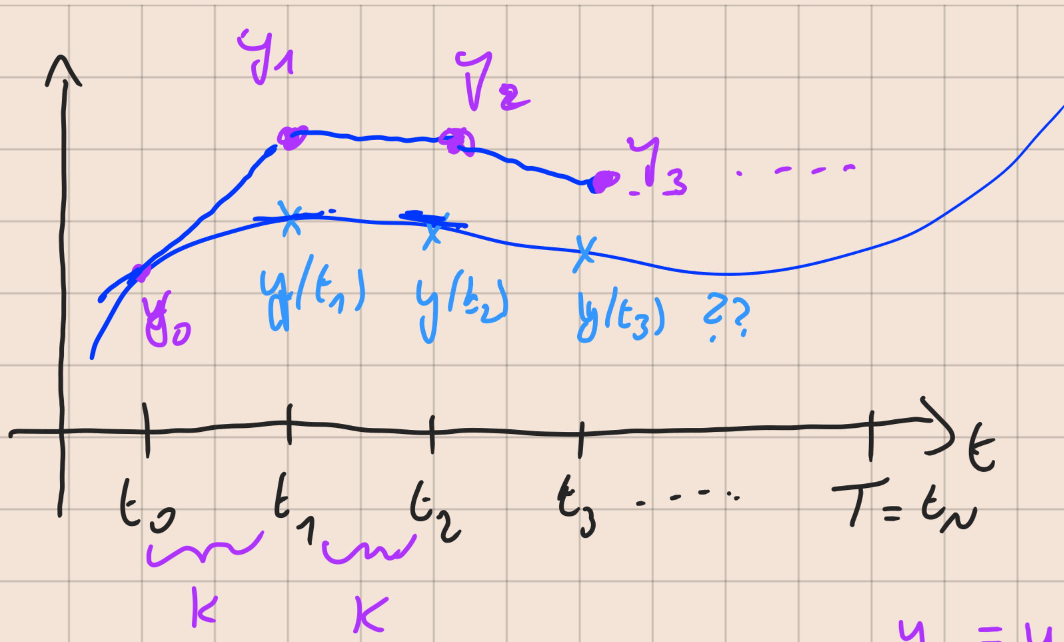
For $t = t_1$: $y(t_2) \approx \underbrace{y(t_1)}_{\approx\, y_1} + k \cdot f(\underbrace{y(t_1)}_{\approx\, y_1}) \Rightarrow$

$$y(t_2) \approx y_1 + k f(y_1) =: y_2 \Rightarrow y_2 \approx y(t_2)$$

$\vdots$

Euler scheme:
$$\begin{cases} y_{n+1} = y_n + k f(y_n) \\ y_0 = y(0) \end{cases} \quad n = 0, 1, 2, \dots, N-1$$

$$\underset{\text{numerical}}{y_n} \approx \underset{\text{exact}}{y(t_n)}$$



$y(t) = ?$

$\dot{y}(t) = f(y(t))$

$y_1 = y_0 + k \cdot f(y_0)$

slope of sol. at (t_0, y_0)

$f(y) = y^2$, $f(1) = 1$, $f(-1) = 1$

$$y_{n+1} = y_n + k f(y_n) \approx y(t_{n+1})$$

$$|y_{n+1} - y(t_{n+1})| \leq C \cdot k^2$$

$k = 0,01$

Similarly, using backward difference, one gets:

Backward Euler scheme: $y_{n+1} = y_n + k \cdot f(y_{n+1})$

Another possibility is to use
Crank-Nicolson scheme

$$y_{n+1} = y_n + \frac{k}{2}\left(f(y_n) + f(y_{n+1})\right)$$

$\hookrightarrow$ $y_n \approx y(t_n)$

Ex: Application of Euler scheme to PDE, see next chapter + lab

Consider the following linear system of DE:

$$M \cdot \dot{\xi}(t) + S \cdot \xi(t) = F(t), \text{ where}$$

$M \leftrightarrow$ mass matrix

$S \leftrightarrow$ stiffness "

$F(t) \leftrightarrow$ load vector

$\zeta(t) \leftarrow$ unknown vector

(i) Let us apply (forward) Euler scheme:

$$M\left(\frac{\zeta^{(n+1)}-\zeta^{(n)}}{k}\right)+S\zeta^{(n)}=F(t_n)$$

Here, k is time step

$\underset{\text{num.}}{\zeta^{(n)}}$ vector $\approx \underset{\text{exact}}{\zeta(t_n)}$

Start $\zeta^{(0)}=\zeta(0)$

$$M\zeta^{(n+1)} = M\zeta^{(n)} - k\cdot S\cdot \zeta^{(n)} + k\cdot F(t_n)$$
$$= (M-k\cdot S)\cdot \zeta^{(n)} + k\cdot F(t_n)$$

Solve this linear system to get $\zeta^{(n+1)}$

(ii) Apply backward Euler scheme:

$$M\left(\frac{\xi^{(n+1)}-\xi^{(n)}}{k}\right)+S'\xi^{(n+1)}=F(t_{n+1})$$

or

$$M\xi^{(n+1)}+kS'\xi^{(n+1)}=M\xi^{(n)}+kF(t_{n+1})$$

or

$$(M+k\cdot S)\,\xi^{(n+1)}=M\xi^{(n)}+k\cdot F(t_{n+1})$$

$$Ax=b$$

(iii) For Crank-Nicolson, one gets

$$\left(M+\frac{k}{2}S'\right)\xi^{(n+1)}=\left(M-\frac{k}{2}S'\right)\xi^{(n)}+\frac{k}{2}\left(F(t_n)+F(t_{n+1})\right)$$

Chapter VII: Partial differential equations in 1d

Goal: Use FEM + "Euler scheme" to find numerical approximations to solution to PDE.

1) Heat equation in 1d:

Model problem to describe heat flow in a thin wire: Heat equation

$$
\begin{cases}
u_t(x,t) - u_{xx}(x,t) = f(x,t) & , \quad 0 < x < 1, \ 0 < t \leq T \quad \text{DE} \\
u(0,t) = 0, \ u_x(1,t) = 0 & , \quad 0 < t \leq T \quad \text{BC} \\
u(x,0) = u_0(x) & , \quad 0 < x < 1, \quad \text{IC}
\end{cases}
$$

$u(x,t) \hookrightarrow$ temperature at point x, at time t

$u_0(x) \longleftrightarrow$ initial temperature profile

$u(0,t) \equiv 0 \leftrightarrow$ hom. Dirichlet BC $\leftrightarrow$

$\leftrightarrow$ fixed 0 temperature at $x \equiv 0$

$u_x(1,t) \equiv 0 \leftrightarrow$ hom. Neumann BC $\leftrightarrow$

$\leftrightarrow$ insulated boundary at $x = 1$

$\leftrightarrow$ no heat flux at $x = 0$

$f(x,t) \leftrightarrow$ heat sources / sink

<u>Th</u>: The sol. to the above heat eq. satisfies the stability estimates

(i) $\|u(\cdot,t)\|_{L^2(0,1)} \leq \|u_0\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\, ds$

(ii) $\|u_x(\cdot,t)\|_{L^2(0,1)} \leq \|u_0'\|_{L^2(0,1)} + \int_0^t \|f(\cdot,s)\|_{L^2(0,1)}\, ds$

(iii) When $f \equiv 0$, one has

$$\underbrace{\|u(\cdot,t)\|_{L^2(0,1)}}_{\sqrt{\int_0^1 |u(x,t)|^2 dx}} \leq \|u_0\|_{L^2(0,1)} \underbrace{e^{-t}}_{t \to \infty}$$

In average, the temperature will go to zero ($e^{-t} \underset{t\to\infty}{\to} 0$) as $t \to \infty$

2) Discretisation of heat equation:

Consider

$$(H)\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & 0<x<1,\ 0<t\leq T \\ u(0,t) = 0 = u(1,t) & 0<t\leq T \\ u(x,0) = u_0(x) & 0<x<1 \end{cases}$$

(simple) hom. Dirichlet BC.

In general, difficult to find the exact sol. $u(x,t)$ ⤳ need to find a numerical approximation!

<u>Idea:</u> VF ⤳ FE ⤳ ODE ⤳ "Euler"

(i) We consider the space

$$V^0 = \{ v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1), \quad v(0) = v(1) = 0 \}$$

hom. Dir. BC

Multiply DE with test function $v \in V^0$, integrate in space, by parts to get:

$$\int_0^1 u_t(x,t) v(x)\,dx - \underbrace{\int_0^1 u_{xx}(x,t) v(x)\,dx}_{\underbrace{u_x(x,t)v(x)\Big|_{x=0}^{1}}_{=0 \text{ since } v \in V^0 \ (v(0)=v(1)=0)} - \int_0^1 u_x(x,t) v_x(x)\,dx} = \int_0^1 f(x,t) v(x)\,dx$$

This gives us the VF: For all $0 < t \leq T$,

(VF) Find $u(\cdot,t) \in V^0$ s.t. $\int_0^1 u_t(x,t) v(x)\,dx + \int_0^1 u_x(x,t) v_x(x)\,dx$

$$= \int_0^1 f(x,t) v(x)\,dx \quad \forall v \in V^0$$

$u(x,0) = u_0(x)$

or ... $(u_t(\cdot,t), v)_{L^2} + (u_x(\cdot,t), v_x)_{L^2} = (f(\cdot,t), v)_{L^2} \quad \forall v$

(ii) Consider subspace of V^0:

$$V_h^0 = \{ v : [0,1] \to \mathbb{R} : v \text{ is cont. pw linear}^{\text{on } T_h}, v(0) = v(1) = 0 \}$$

$$= \text{span}(\varphi_1, \varphi_2, \dots, \varphi_m) \quad , \text{where}$$

φ_j are hat functions, $h = \frac{1}{m+1}$,

$T_h : 0 = x_0 < x_1 < x_2 < \dots < x_{m+1} = 1.$

We then get the FE problem,

for $0 < t \leq T$

(FE) $\begin{cases} \text{Find } U(\cdot,t) \in V_h^0 \text{ s.t.} \\ (U_t(\cdot,t), \chi)_{L^2} + (U_x(\cdot,t), \chi_x)_{L^2} = \underbrace{(f(\cdot,t),\chi)_{L^2}}_{= \int_0^1 f(x,t)\chi(x)\,dx} \quad \forall \chi \in V_h^0 \\ U(x,0) = \underbrace{\Pi_h u_0(x)}_{\text{pw linear interpolant of } u_0.} \end{cases}$