Recall: ODE/IVP $\begin{cases} \dot{y}(t) = f(y(t)) & 0 < t \leq T \\ y(0) = y_0 & \text{IC} \end{cases}$

BVP $\begin{cases} u''(x) - u(x) = g(x) & 0 < x < 1 \\ u(0) = 0 = u(1) & \text{BC} \end{cases}$ FEM $U(x) \approx u(x)$

PDE $\begin{cases} u_t(t,x) - u_{xx}(t,x) = h(t,x), & 0 < t \leq T, \ 0 < x < 1 \\ + \text{BC} \\ + \text{IC} \end{cases}$

Chapter VI: Scalar initial value problem (IVP)

Goal: Study and numerically approximate solutions to IVP

$$(\text{IVP}) \begin{cases} \dot{y}(t) = f(y(t)) & 0 < t \leq T \\ y(0) = y_0, \end{cases}$$

?

where $T > 0$, y_0, $f : \mathbb{R} \to \mathbb{R}$ are given,

$$\dot{y}(t) = \frac{d}{dt} y(t)$$

1) Linear first order IVPs:

We consider the model problem

$$(\text{IVP}) \quad \begin{cases} \dot{y}(t) + a(t)\,y(t) = f(t) & 0 < t \le T \\ y(0) = y_0, \end{cases}$$

here $y_0 \in \mathbb{R}$ given, f given (bounded, continuous),

$a(t)$ given (continuous, $a(t) \geq 0$ and bounded)

Ex: Simple model in population dynamics, see chapter I

$$\dot{y}(t) = a\,y(t), \quad a \in \mathbb{R}$$

Th: The solution to the above IVP is given by the variation of constant formula (VOC):

$$y(t) = e^{-A(t)} y_0 + \int_0^t e^{-(A(t)-A(s))} f(s)\,ds,$$

where $A(t) := \int_0^t a(s)\,ds$. (a(s) constant ~ easy)

Proof:

- Multiply the DE with the integrating factor $e^{A(t)}$:

$$\underbrace{\dot{y}(t)\,e^{A(t)} + a(t)\,e^{A(t)}\,y(t)}_{= \frac{d}{dt}\left(y(t)\,e^{A(t)}\right)} = f(t)\,e^{A(t)}$$

$$= \frac{d}{dt}\left(y(t)\,e^{A(t)}\right) = \dot{y}(t)\,e^{A(t)} + y(t)\,\underbrace{\frac{d}{dt}\left(e^{A(t)}\right)}_{= e^{A(t)} \frac{d}{dt} A(t) = e^{A(t)} \cdot a(t)}$$

$a(t)$

product rule

- We integrate the expression $\frac{d}{dt}\left(y(t)e^{A(t)}\right) = f(t)e^{A(t)}$

and get $y(t)e^{A(t)} - \underbrace{y(0)e^{A(0)}}_{y_0 \cdot 1} = \int_0^t f(s)e^{A(s)}\,ds$

since $A(0)=0$, $e^{A(0)} = e^0 = 1$

Then,

$$y(t) = e^{-A(t)} y_0 + \int_0^t e^{-(A(t)-A(s))} f(s)\,ds \quad :-)$$

$\blacksquare$

We next look at the stability of the sol. to this IVP

<u>Th:</u> Let $y(t)$ denotes the above VOC (i.e. sol. to (IVP)). Then, we have

(dissipative case)

(i) If $a(t) \geq \alpha > 0$, then

$$|y(t)| \leq e^{-\alpha t}|y_0| + \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\max_{0\leq s\leq t}|f(s)|$$

(parabolic case)

(ii) If $a(t) \geq 0$, then

$$|y(t)| \leq |y_0| + \int_0^t |f(s)|\,ds$$

Proof:

(i) • Since $a(t) \geq \alpha > 0$, then $\int_0^t a(s)\,ds \geq \int_0^t \alpha\,ds$ or

$A(t) \geq \alpha t$ by def of $A(t)$.

Then, $-A(t) \leq -\alpha t$ or $e^{-A(t)} \leq e^{-\alpha t}$

Similarly, one has $e^{-(A(t)-A(s))} \leq e^{-\alpha(t-s)}$.

• From the voc, we thus get

$$|y(t)| \overset{\Delta}{\leq} \underbrace{e^{-A(t)}}_{\leq e^{-\alpha t}} |y_0| + \int_0^t \underbrace{e^{-(A(t)-A(s))}}_{\alpha} \underbrace{|f(s)|}_{\leq \max_{0\leq r\leq t} |f(r)|.} ds \quad (*)$$

$$\leq e^{-\alpha t} |y_0| + \max_{0\leq s\leq t} |f(s)| \cdot \underbrace{\int_0^t e^{-\alpha(t-s)} ds}_{\left.\frac{e^{-\alpha(t-s)}}{\alpha}\right|_{s=0}^{t}}$$

$$\leq e^{-\alpha t} |y_0| + \max_{0\leq s\leq t} |f(s)| \cdot \left(\frac{1}{\alpha} - \frac{e^{-\alpha t}}{\alpha} \right).$$

(ii) $a(s) \geq 0 \Rightarrow A(t) \geq 0 \Rightarrow -A(t) \leq 0 \Rightarrow e^{-A(t)} \leq 1$

Then, we get $|y(t)| \overset{\Delta}{\leq} 1 \cdot |y_0| + \int_0^t 1 \cdot |f(s)|\,ds$

[Hampus:

indep. of s!

$$\int_0^t |f(s)|\,ds \le \int_0^t \underbrace{\max_{0\le s\le t} |f(s)|}\,ds \le \max_{0\le \pi\le t}|f(\pi)| \int_0^t ds$$

$$\int_0^t |\cos(22\cdot s)|\,ds \le \int_0^t \underbrace{\max_{0\le s\le t} |\cos(22 s)|}_{\le 1}\,ds \le$$

$\le$

]

Ex. Find and plot the solutions to IVP

(i) $\begin{cases} y'(t) + \overset{=a(t)}{2}\, y(t) = \underset{= f(t)}{t} & , 0 < t \le 3 \\ y(0) = \frac{3}{4} \end{cases}$ $a(t) = 2 = \alpha > 0$ (dissipative)

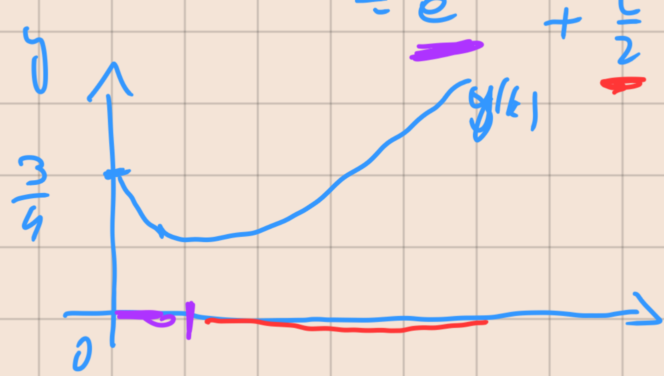
(ii) $\begin{cases} y'(t) = t & , 0 \le t \le 3 \\ y(0) = \frac{3}{4} \end{cases}$ $a(t) = 0$ (parabolic)

(i) We have $A(t) = \int_0^t 2\,ds = 2t$, and the voc gives

$$y(t) = e^{-2t} y(0) + \int_0^t e^{-(2t-2s)} s\,ds \overset{\text{by parts}}{=} \dots$$

$$= e^{-2t}\frac{3}{4} + \frac{1}{2}t - \frac{1}{4}\left(1 - e^{-2t}\right) =$$

$$= e^{-2t} + \frac{t}{2} - \frac{1}{4}$$



Th: voc $y(t) = e^{-A(t)} y(0) + \dots$
$=$

Th: $|y(t)| \le e^{-\alpha t} y(0) + \dots$

(ii) $\begin{cases} \dot{y}(t) = t \\ y(0) = \frac{3}{4} \end{cases}$ $\rightsquigarrow$ solve it directly by integrating

$$y(t) = \frac{t^2}{2} + \text{const} \underset{\substack{\uparrow \\ IC}}{=} \frac{t^2}{2} + \frac{3}{4}$$

2) Finite difference:

Consider $y : \mathbb{R} \to \mathbb{R}$ and $t_0 \in \mathbb{R}$. The fct y is differentiable in t_0 if

$$y'(t_0) = \dot{y}(t_0) = \lim_{h \to 0} \frac{y(t_0+h) - y(t_0)}{h}$$

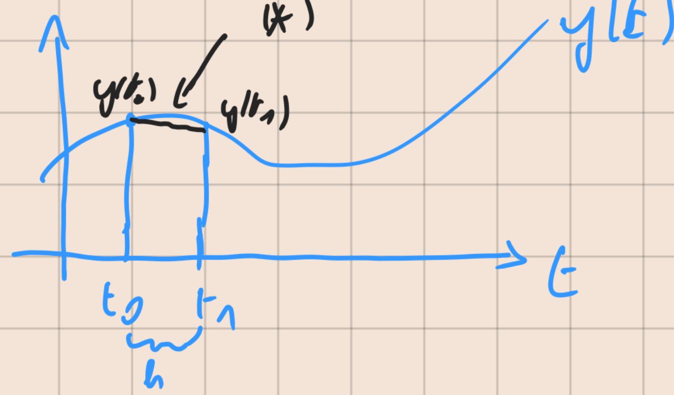
<u>Idea/Defi</u> For a fixed (small) $h > 0$, one gets an approximation of the derivative of y as follows

$$\dot{y}(t_0) \approx \frac{y(t_0+h) - y(t_0)}{h} \underset{\substack{\uparrow \\ t_1 = t_0 + h}}{=} \frac{y(t_1) - y(t_0)}{t_1 - t_0}$$

This is called a <u>forward difference</u>.

(*) Slope of black line

$$\frac{y(t_1)-y(t_0)}{t_1-t_0}$$



Similarly, one can define the following approximations

$$\dot y(t_0) \approx \frac{y(t_0)-y(t_0-h)}{h}$$ backward difference

$$\dot y(t_0) \approx \frac{y(t_0+h)-y(t_0-h)}{2h}$$ centered difference

3) First time integrators for IVP:

Consider

$$(IVP) \quad \begin{cases} \dot y(t) = f(y(t)) & 0<t\leq T \\ y(0) = y_0 \end{cases}$$

Let $N\in\mathbb{N}$ (BIG), define time step $k=\frac{T}{N}$ and consider a grid: $0=t_0<t_1<t_2<\dots<t_N=T$,

where $t_n = n\cdot k$ for $n=0,1,2,\dots,N$

Idea (Euler, 1768)

For $t_0 \le t \le t_1 = t_0 + k$, we replace $\dot{y}(t)$ by a forward difference:

$$\dot{y}(t) = f(y(t)) \rightsquigarrow \frac{y(t+k) - y(t)}{k} \approx f(y(t))$$

or $\quad y(t+k) \approx y(t) + k \cdot f(y(t))$

Take $t = t_0$ above and get

$$y(t_0 + k) \approx y(t_0) + k \cdot f(y(t_0)) \quad \text{or}$$

$$\underbrace{y(t_1)}_{\substack{\text{exact} \\ \text{sol. to IVP} \\ \text{(unknown)}}} \approx \underbrace{y_0 + k \cdot f(y_0)}_{\text{numerical approximation}} = y_1$$

Using this formula recursively, one gets <u>Euler scheme</u>:

$$\begin{cases} y_0 = y(t_0) \\ y_{n+1} = y_n + k \cdot f(y_n) \quad \text{for } n = 0, 1, \ldots, N-1 \end{cases}$$

and obtain num. approx

$$y_n \approx y(t_n)$$

numerical

exact