Recall:

Inhomogeneous heat eq.

$$\begin{cases} u_t - u_{xx} = 8 \\ u(0,t) = 1,\ u(1,t) = 3 \\ u(x,0) = f(x) \end{cases}$$

[or $8x$ or $g(x)$]

(i) $u(x,t) = v(x,t) + s(x)$

Into PDE: $\underline{v_t - v_{xx}} - \underline{s''} = \underline{8} + \underline{0}$ ...

Gives BVP $\begin{cases} -s''(x) = 8 \\ s(0) = 1,\ s(1) = 8 \end{cases}$ PDE $\begin{cases} v_t(x,t) - v_{xx}(x,t) = 0 \\ v(0,t) = 0 = v(1,t) \\ v(x,0) = f(x) - s(x) \end{cases}$

(ii) Solve BVP

Solve PDE using sep. of variables $v(x,t) = X(x) \cdot T(t)$

a) ODE for $T(t)$

b) BVP for $X(t)$ (+ BC)

c) Superposition $v(x,t) = \sum_{n=1} A_n T_n(t) X_n(x)$

d) IC + Fourier series to get A_n

LAST EXAM on CANVAS (OMTENTA 08/20)

1. Använd Laplacetransformer för att lösa differentialekvationen

$$y''(t) + 2y'(t) + 3y(t) = 3t, \qquad y(0) = 0, \quad y'(0) = 1.$$

① Take LT of DE: Noting $Y(s) = \mathcal{L}\{y\}(s)$, one gets

linearity of LT

Table

$$\mathcal{L}\{y''\}(s) + 2\mathcal{L}\{y'\}(s) + 3Y(s) = \frac{3}{s^2}$$

$$\mathcal{L}\{y'\}(s) = sY(s) - y(0) = sY(s) + 0$$

Table

$\vdots$

$$(s^2 + 2s + 3)Y(s) = 1 + \frac{3}{s^2}$$

$\vdots$

$$Y(s) = \frac{s^2 + 3}{\underbrace{s^2(s^2 + 2s + 3)}}$$

② Write the term as "simpler blocks":

$$\frac{s^2 + 3}{s^2((s+1)^2 + 2)} \stackrel{\text{partial fractions}}{=} \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{((s+1)^2 + 2)}$$

$(s-0)^2$

Find A, B, C, D by solving a linear system ....

$$A = -\frac{2}{3}, \quad B = 1, \quad C = \frac{2}{3}, \quad D = \frac{4}{3}$$

(2) Get $Y(s) = -\frac{2}{3}\frac{1}{s} + \frac{1}{s^2} + \frac{2}{3}\frac{s+1+1}{(s+1)^2+2}$

③ Take ILT of above to get:

$$y(t) = \mathcal{L}^{-1}\{Y\}(t) = -\frac{2}{3} + t + \frac{2}{3}e^{-t}\cos(\sqrt{2}t) + \frac{\sqrt{2}}{3}e^{-t}\sin(\sqrt{2}t)$$

Table

$$\frac{s+1}{(s+1)^2+(\sqrt{2})^2} \sim \frac{\hat{s}}{(\hat{s})^2+(\sqrt{2})^2}$$

5. Härled *variationsformulering* för begynnelsevärdesproblemet (a, b, α, β är icke-nollkonstanter),

$$\begin{cases} -u'' + au' + bu = f, \quad 0 < x < 1, \\ u'(0) = \alpha, \quad u(1) = \beta. \end{cases}$$

(i) Multiply DE by a test function v, integrate, by parts:

$$\underbrace{-\int_0^1 u''(x)v(x)dx}_{-u'(1)v(1) + u'(0)v(0) + \int_0^1 u'(x)v'(x)dx} + a\int_0^1 u'(x)v(x)dx + b\int_0^1 u(x)v(x) = \int_0^1 f(x)v(x)dx$$

$-u'(1)v(1) + u'(0)v(0) + \int_0^1 u'(x)v'(x)dx$

$u'(1)v(1) \stackrel{!}{=} 0$, $u'(0) = \alpha$ (BC)

by imposing $v(1) \equiv 0 \rightarrow$ this will be inserted in the test space

$u(1) = \beta$

(ii) Define test space $\tilde{V} = \{v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1) \text{ and } v(1) = 0\}$

Trial space $V = \{v : [0,1] \to \mathbb{R} : v, v' \in L^2(0,1) \text{ and } v(1) = \beta\}$.

inh. Dirich.

(VF) Find $u \in V$ s.t.

$$\int_0^1 u'v'dx + a\int_0^1 u'vdx + b\int_0^1 uvdx = \int_0^1 fvdx - \overset{u'(0)}{\cancel{\alpha}}\, v(0)$$

$\forall v \in \tilde{V}$.

3. (a) För vilka värden på $a > 0$ är funktionerna $\sin(ax)$ och $\cos(ax)$ ortogonala i $L_2(0,1)$? (4p)

(b) Bestäm Fourier cosine-serien med perioden 2π till funktionen $f(x) = \sin x,\ 0 \leq x \leq \pi$. (5p)

4. Betrakta den inhomogena vågekvationen (10p)

$$\begin{cases} \ddot{u}(x,t) - u''(x,t) = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 0, \ u(1,t) = 1, & \qquad\qquad\quad t > 0, \\ u(x,0) = 2x & 0 < x < 1 \\ \dot{u}(x,0) = 0 & 0 < x < 1 \end{cases}$$

Använd variabelseparationsmetoden för att bestämma $u(x,t)$.

6. Betrakta begynnelsevärdesproblemet

$$\begin{cases} -u'' + 2u = 3, \quad 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

(a) Härled *variationsformulering.* (3p)

(b) Härled cG(1) finita element formulering (kontinuerliga styckvis linjära polynomer). Härled det linjära ekvationssystemet på formen $S\xi + M\xi = F$. Beräkna styvhetsmatrisen S (Stiffness matrix) och lastvektorn F (Load vector). Beräkna **ej** massmatrisen M (Mass matrix). (7p)

(OBS! Använd likformig partition med steglängd h och $\mathcal{T}_h : \ 0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = 1$.)