Recall: • f 2π-periodic and integrable on $[-\pi,\pi]$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

$$\sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$\left(e^{ix} = \cos(x) + i\sin(x)\right)$

Fourier series,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n = 0,1,2,\ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad n = 1,2,3,\ldots$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

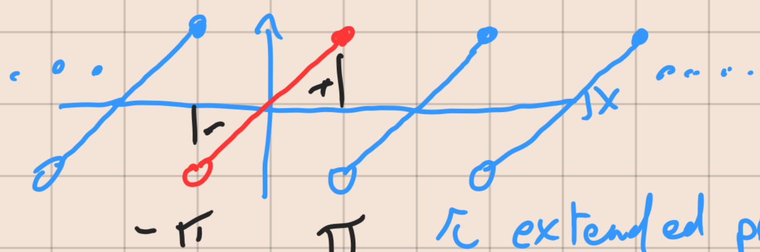
$n \in \mathbb{Z}$,
$n = 0, \pm 1, \pm 2, \pm 3, \ldots$

• $$\int_{-\pi}^{\pi} e^{inx} e^{-imx}\,dx = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

⚠

$L^2(-\pi,\pi)$

Ex: FS of $f(x) = x$ for $-\pi < x \le \pi$



extended periodically

Compute Fourier coeff.:

$$a_0 \underset{\text{Def}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \underset{\text{Def } f}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0 \qquad \frac{1}{\pi}\left(\frac{x^2}{2}\Big|_{-\pi}^{\pi}\right) = 0$$

$$a_n \underset{\text{Def } a_n}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \underset{\text{Def } f}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{v}{x}\,\underset{u'}{\cos(nx)}\,dx =$$

$$\underset{\text{by part}}{=} \frac{1}{\pi}\left(x\,\frac{\sin(nx)}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin(nx)}{n}\,dx \right) =$$

$$= \frac{1}{\pi}\left(\frac{\pi\sin(n\pi) - (-\pi)\sin(n(-\pi))}{n} + \frac{1}{n}\,\frac{\cos(nx)}{n}\Big|_{-\pi}^{\pi} \right)$$

$$= \frac{1}{\pi}\left(0 + \frac{1}{n^2}\left(\cos(n\pi) - \cos(n(-\pi))\right)\right) =$$

$$= \frac{1}{\pi n^2}\left(\cos(n\pi) - \cos(n\pi)\right) = 0 \quad \forall n$$

$$b_n \overset{\text{Def}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \overset{\text{Def } f}{=}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{v}{x}\,\underset{u'}{\sin(nx)}\,dx \overset{\text{by part}}{=}$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}x\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx\right)$$

$$\Big| \; \cos(-\alpha) = \cos(\alpha)$$

$$= \frac{1}{\pi}\left(-\frac{1}{n}\left(\pi\cos(n\pi) - (-\pi)\cos(n\pi)\right)\right.$$

$$\left. + \frac{1}{n}\,\frac{\sin(nx)}{n}\Big|_{-\pi}^{\pi}\right) =$$

$$= -\frac{1}{n}\cdot 2\cdot(-1)^n = 2\cdot\frac{(-1)^{n+1}}{n} \quad \text{for } n = 1, 2, \ldots$$

$\hookrightarrow$ The Fourier series of f reads

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) \quad //$$

Rem: $a_n = 0$ for $n = 0, 1, 2, \dots$ since f is odd!

4) Even and odd functions:

Def: • A function f is called even if $f(-x) = f(x) \quad \forall x$

• A function f is called odd if $f(-x) = -f(x) \quad \forall x$

Ex: x^2, $\cos(x)$ are even fct.

x, $\sin(x)$ are odd fct.

Rem: One has, for any $a \in \mathbb{R}$,

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f \text{ odd} \\ 2\int_0^a f(x)\,dx & \text{if } f \text{ even} \end{cases}$$

(*)

Rem: even $\times$ even $\rightarrow$ even

multiplication

even $\times$ odd $\rightsquigarrow$ odd

odd $\times$ odd $\rightsquigarrow$ even

With the above, one can show

Lemma:

For a even function f, one has

$$\underbrace{a_n}_{\text{Def } a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{f(x)}_{\text{even}}\underbrace{\cos(nx)}_{\text{even}}}_{\text{even}}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin(nx)}_{\text{odd}}\,dx = 0 \;!!$$

Def b_n

even odd

For an odd fct f, one has

$a_n = 0$ for $n = 0, 1, 2, \dots$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx \quad \text{for } n = 1, 2, \dots$$

(*) f even $\to$ $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx +$

$x = -y$
$dx = -dy$

$$+\int_0^a f(x)\,dx = \underbrace{\int_a^0 f(-y)\,dy}_{} + \int_0^a f(x)\,dx$$

$$\int_0^a f(-y)\,dy = \int_0^a f(y)\,dy$$

f even

$$= 2\int_0^a f(x)\,dx$$

Rem:

- f odd $\Rightarrow a_n = 0 \Rightarrow$ FS has only terms with $\sin(nx) \Rightarrow$ sine-Fourier series
- f even $\Rightarrow b_n = 0 \Rightarrow$ FS has only terms with $\cos(nx) \Rightarrow$ cosine-Fourier series

5) Bessel's inequality and Riemann-Lebesgue lemma:

Intermezzo:

$\mathbb{R}^d$ $\qquad \|x\| = \sqrt{(x,x)} \qquad (x,y) = x^T y = \sum_j x_j y_j$

$$e_j = (0, 0, \dots, 0, \underset{j}{1}, 0, \dots, 0)$$

$$\mathbb{R}^d \ni x = \sum_{n=1}^{d} x_n e_n \qquad x_n \sim (x, e_n)$$

$L^2(-\pi,\pi)$ $\qquad (f,g) = \int_{-\pi}^{\pi} f(x)\overline{g}(x)\,dx$

e^{inx}

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$\overset{?}{=}$

$c_n \sim (f, e^{inx})$

<u>Th</u>: (Bessel's inequality)

Let $f \in L^2(-\pi,\pi)$ be 2π-periodic.

Then

$$\sum_{n=-\infty}^{\infty} \underbrace{|c_n|^2}_{c_n\overline{c_n}} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

<u>Proof</u>:

- Let $N \in \mathbb{N}$ be fixed and consider

$$0 \leq \left| f(x) - \sum_{n=-N}^{N} c_n e^{inx} \right|^2 =$$

modulus ≥ 0

$|z|^2 = z\bar{z}$

$$= \left(f(x) - \sum_{n=-N}^{N} c_n e^{inx} \right) \left(\overline{f(x)} - \sum_{n=-N}^{N} \bar{c}_n e^{-inx} \right)$$

$$= f(x)\overline{f(x)} - \sum_{n=-N}^{N} c_n e^{inx} \overline{f(x)} - \sum_{n=-N}^{N} \bar{c}_n f(x) e^{-inx}$$

$$+ \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_n \bar{c}_m e^{inx} e^{-imx}$$

- We integrate $\int_{-\pi}^{\pi}$:

$$0 \leq \int_{-\pi}^{\pi} \underbrace{f(x)\overline{f(x)}}_{|f(x)|^2} dx - \sum_{n=-N}^{N} c_n \underbrace{\int_{-\pi}^{\pi} \overline{f(x)} e^{inx} dx}_{2\pi \overline{c_n}}$$

$$- \sum_{n=-N}^{N} \bar{c}_n \underbrace{\int_{-\pi}^{\pi} f(x) e^{-inx} dx}_{2\pi c_n}$$

by Def of c_n

$$+\sum_n \sum_m c_n \bar{c}_m \underbrace{\int_{-\pi}^{\pi} e^{inx} e^{-imx}\,dx}_{\substack{=0 \text{ if } n\neq m \\ =2\pi \text{ if } n=m}}$$

We then get

$$0 \le \int_{-\pi}^{\pi} |f(x)|^2\,dx - 2\pi \sum_{n=-N}^{N} |c_n|^2 - 2\pi \sum_{n=-N}^{N} |c_n|^2$$

$$+ 2\pi \sum_{n=-N}^{N} |c_n|^2$$

Finally,

$$2\pi \sum_{n=-N}^{N} |c_n|^2 \le \int_{-\pi}^{\pi} |f(x)|^2\,dx \quad \text{or}$$

$$\sum_{n=-N}^{N} |c_n|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\,dx \qquad \forall N$$

Taking $N \to \infty$ shows that $\sum_{n=-\infty}^{\infty} |c_n|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\,dx$

:-)

$\blacksquare$

__Lemma__ (Riemann-Lebesgue)

Under the assumptions of Bessel, the Fourier coefficients (a_n, b_n, or c_n) converge to zero as n goes to infinity,

$$c_n \xrightarrow[n \to \pm\infty]{} 0$$

$$b_n, a_n \xrightarrow[n \to \infty]{} 0$$

__Proof:__

Using Bessel, one gets

$\sum_{n=-\infty}^{\infty} |c_n|^2$ converges, then

$|c_n|^2$ must go to zero as $n \to \pm\infty$,

then so does c_n go to zero.

Then a_n and b_n are also going to zero as $n \to \infty$ ☒

Why should we care?

Rem: If a_n, b_n, or $c_n \underset{n\to\infty}{\not\to} 0$

then the Fourier series cannot converge.

6) Convergence of Fourier series:

Recall: A series $\sum_{n=1}^{\infty} x_n$ converges to say $x \in \mathbb{R}$, if the partial sums $\sum_{n=1}^{N} x_n$ converges as $N \to \infty$

$$\lim_{N\to\infty} \left(\sum_{n=1}^{N} x_n \right) = x.$$

Th (Dirichlet Theorem)

Let f be 2π-periodic with f and f' piecewise continuous.

Set
$$S_N^f(x) := \sum_{n=-N}^{N} c_n e^{inx}, \text{ where}$$

c_n are the Fourier coeff. of f.

Then, we have

$$\lim_{N\to\infty} S_N^f(x_0) = \frac{f(x_0^-) + f(x_0^+)}{2} \quad \forall x_0$$

If f is continuous at x_0, then

$$\lim_{N\to\infty} S_N^f(x_0) = f(x_0). \quad \left(\sum_{n=-\infty}^{\infty} c_n e^{inx_0} = f(x_0) \right)$$

Recall: $f(x_0^-) = \lim_{x\to x_0^-} f(x)$

(limit from left)

Th (Parseval identity)

For f 2π-periodic and in $L^2(-\pi,\pi)$, one has

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

Rem: The above may help to compute limits of series!