Chapter I: Introduction and motivation

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Terminology:

- A differential equation (DE) is an equation that relates an unknown fct and its derivative(s).

- An ordinary differential equation (ODE) is a DE, where the unknown fct depends only on 1 variable.
  "fct ⇝ function" [$y(x), x(t)$, …

- A partial differential equation (PDE), is a DE, where the unknown fct depends on 2 or more variables ($u(x,y)$, $u(t,x,y)$ …

Ex: (ODE)

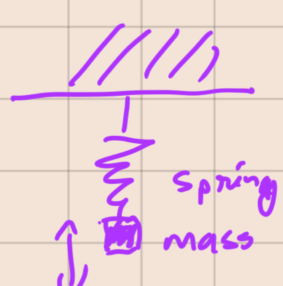
(i) Malthusian growth model for bacteria reads:

$$\frac{d}{dt} P(t) = \lambda \cdot P(t).$$

Here λ given
$P(t)$ ⇝ size of population at time t.

Notation: $\dot{P}(t) = \frac{d}{dt} P(t)$

(ii) Mass-spring system:



Newton's law: $\underset{\substack{\text{mass}\\\text{(given)}}}{m} \cdot \underbrace{\frac{d^2}{dt^2} x(t)}_{\text{acceleration}} + \underbrace{k \cdot x(t)}_{\text{Hook's law}} = 0$ ← given

x = ?? unknown fct depending on 1 variable (t)

(iii) $u'(x) + 5u(x) + \cos(x) \cdot u(x)^3 = 0$

u: unknown, depends on "x".

(iv), .. etc.

To determine a unique solution to an ODE, one needs additional conditions.

Ex (IVP)

Adding an initial condition/initial value gives an initial value problem (IVP)

$$\begin{cases} \frac{d}{dt} P(t) = \lambda P(t) \\ P(0) = P_0 \end{cases}$$ ← initial condition

$$\begin{cases} \frac{d}{dt} P(t) = 2 \cdot P(t) \\ P(0) = 5 \end{cases}$$

↑ one expl. with $\lambda = 2$, $P_0 = 5$

Ex (BVP)

Add boundary conditions gives a boundary value problem (BVP)

$$\begin{cases} -u''(x) = \cos(x) \quad \text{for } x \in (0,1) \\ u(0) = 0, \; u(1) = 5 \end{cases}$$ ← BC (specify values of u at boundary)

Next, we define Laplace operator (Δ):

(1d) : $\Delta u(x) = u''(x) = u_{xx}(x)$

(2d) : $\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = u_{xx} + u_{yy}$

(3d) : $\Delta u(x,y,z) := \frac{\partial^2 u}{\partial x^2}(x,y,z) + \frac{\partial^2 u}{\partial y^2}(x,y,z) + \frac{\partial^2 u}{\partial z^2}(x,y,z)$

$= u_{xx} + u_{yy} + u_{zz}$

etc.

Ex: Compute $\Delta u(x,y)$ for $u(x,y) = 5x^2 \cdot y^3$

$u_x(x,y) = 10xy^3 \rightarrow u_{xx}(x,y) = 10y^3$

$u_y(x,y) = 15x^2y^2 \rightsquigarrow u_{yy}(x,y) = 30x^2y$

$\Rightarrow \Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y) = 10y^3 + 30x^2y$

Ex (PDE)

(i) Laplace equation $\Delta u = 0$

(1d: Find $u(x)$ s.t. $u''(x) = 0$)

(ii) Heat equation $u_t - \Delta u = f$

(1d: Find $u(x,t)$ s.t. $u_t(x,t) - u_{xx}(x,t) = f(x)$)

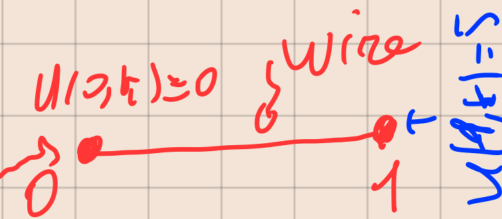
(iii) Wave equation $u_{tt} - \Delta u = g$

(1d: $u_{tt}(x,t) - u_{xx}(\cancel{x},t) = g(x)$)

Here, we again add conditions to find a unique solution

Ex: Heat in $[0,1]$

given $u(0,t)=0$



$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x) \\ \boxed{u(0,t)=0},\ \boxed{u(1,t)=5} \\ u(x,0) = \underbrace{u_0(x)}_{\text{given initial value (function)}} \end{cases}$$

B.C. (gives temp. at 0 and 1)

I.C.

$u(x,t) \rightsquigarrow$ temperature at point $x \in [0,1]$ at time t

Chapter II: Mathematical tools

Goal: Define (abstract) spaces, where we will live and work.

1) Vector spaces

Rem: If vectors in $\mathbb{R}^2$ OK $\Rightarrow$ rest is OK

Def: A set V of vectors or functions is called a vector space or a linear space if $\forall\, u, v, w \in V$ and $\forall\, \alpha, \beta \in \mathbb{R}$, one has:

(i) $u + \alpha \cdot v \in V$

(ii) $(u+v)+w = u+(v+w) = u+v+w$

(iii) $\exists\, 0 \in V$ s.t. $u+0 = 0+u = u \quad \forall u$

(iv) $\forall u \in V$, $\exists\, (-u) \in V$ s.t. $u+(-u) = (-u)+u = 0$

(v) $u+v = v+u$

(vi) $(\alpha+\beta)u = \alpha u + \beta u$

(vii) $\alpha(u+v) = \alpha u + \alpha v$

(viii) $(\alpha\beta)u = \alpha(\beta u) = \alpha\beta u$

$\forall$ ⇝ for all, $\exists$ ⇝ there exists

Ex: • $V = \mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ ← Clear?

← vector in 2d

Add vectors: $(x_1, y_1) + (x_2, y_2) =$

$= (x_1 + x_2, y_1 + y_2)$

Multiply with real number: $\alpha \cdot (x,y) =$

$= (\alpha x, \alpha y)$

$V = \mathbb{R}^2$ is a vector space

$V = \mathbb{R}^3$ or $V = \mathbb{R}$ or $V = \mathbb{R}^{22}$, or

$V = \mathbb{R}^n$ ⇝ same!!

Ex: $V = \mathcal{P}^{(1)}(\mathbb{R}) = \{$ set of polynomials on $\mathbb{R}$ of degree $\leq 1 \}$

$= \{a + bx : x \in \mathbb{R}, a, b \in \mathbb{R}\}$

Add polynomials OK

Multiply by real numbers OK

($\alpha \cdot (a + bx) = \alpha a + \alpha b x$ ⇝ polyn. of degree ≤ 1)

$V = \mathcal{P}^{(1)}(\mathbb{R})$ is a vector space

We can do the same for

$\mathcal{P}^{(2)}(\mathbb{R})$, or $\mathcal{P}^{(27)}(\mathbb{R})$, or $\mathcal{P}^{(n)}(\mathbb{R}) =$

$= \{$ set of polyn. of degree $\leq n \}$