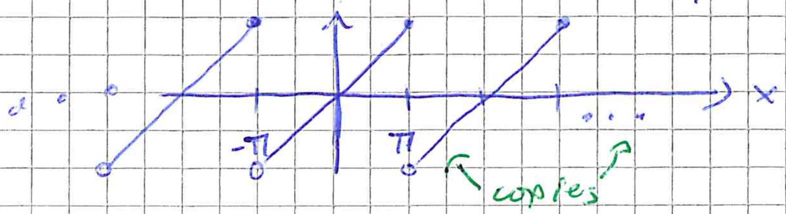
Ex: Find the FS (in a_n, b_m) for $f(x) = x$ with $-\pi < x \leq \pi$ extended periodically ($p = 2\pi$)



Use the definitions:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{\pi}\frac{x^2}{2}\Big|_{-\pi}^{\pi} = 0$$

def a_n def f

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{v}{x}\,\underset{u'}{\cos(nx)}\,dx \overset{\text{by parts}}{=}$$

$u = \frac{1}{n}\sin(nx)$

$$= \frac{1}{\pi}\left(\frac{1}{n}x\sin(nx)\Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi}\sin(nx)\,dx\right) = \dots = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \underset{\text{by parts}}{=} \dots = \frac{1}{\pi}\left(-\frac{1}{n}x\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos(nx)\,dx\right)$$

$$= -\frac{2\pi}{n\pi}(-1)^n - \frac{1}{n^2\pi}\underbrace{\left(\sin(n\pi) - \sin(-n\pi)\right)}_{=0} = (-1)^{n+1}\frac{2}{n}$$

Hence, $f(x) \sim \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin(nx)$

Rem: In above expl, f is an odd fct $\to a_n = 0$ for $n = 0, 1, 2, \dots$

4) Even and odd functions:

Def: A fct f is even if $f(x) = f(-x)$ $\forall x$
A fct f is odd if $f(-x) = -f(x)$ $\forall x$

Ex: $\cos(x)$ is even, $\sin(x)$ is odd.

Rem: even · even $\leadsto$ even || even · odd $\leadsto$ odd || odd · odd $\leadsto$ even

Rem: $\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 & \text{if } f \text{ is odd} \\ 2\int_0^a f(x)\,dx & \text{if } f \text{ is even} \end{cases}$

This permits to show:

Lemma:

If f is even, then $\underset{\text{Def Fourier coeff}}{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{ev.}{f(x)} \underset{ev.}{\cos(nx)}dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx$ for $n=0,1,2,\dots$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{ev.}{f(x)} \underset{od.}{\sin(nx)}dx = 0$ for $n=1,2,\dots$

If f is odd, then $a_n = 0$ and $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$ for $n=1,2,\dots$

Rem: f even $\Rightarrow$ FS has only cosine terms $\to$ Fourier cosine series

f odd $\Rightarrow$ " " sine terms $\to$ Fourier sine series

5) Bessel's inequality and Riemann-Lebesgue lemma:

(P) Th (Bessel's ineq.)

Let $f \in L^2(-\pi,\pi)$ be 2π-periodic. Then,

$$\sum_{n=-\infty}^{\infty} \underbrace{|c_n|^2}_{\left(\sqrt{c_n\bar{c}_n}\right)^2} \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx$$

Proof: Let $N \in \mathbb{N}$ fixed. Consider

$$0 \leq \left|f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right|^2 \overset{\text{def } |\cdot|}{=} \left(f(x) - \sum_{m=-N}^{N} c_m e^{imx}\right)\left(\bar{f}(x) - \sum_{n=-N}^{N} \bar{c}_n e^{-inx}\right) =$$

$$= |f(x)|^2 - \sum_{n=-N}^{N} c_n \bar{f}(x) e^{inx} - \sum_{n=-N}^{N} \bar{c}_n f(x) e^{-inx} + \sum_{n,m=-N}^{N} c_n \bar{c}_m e^{inx} e^{-imx}$$

Integrating $\int_{-\pi}^{\pi}$ the above gives:

$$0 \leq \int_{-\pi}^{\pi} \left|f(x) - \sum_{n=-N}^{N} c_n e^{inx}\right|^2 dx = \int_{-\pi}^{\pi} |f(x)|^2 dx - \sum_n c_n \overbrace{\int_{-\pi}^{\pi} \bar{f}(x) e^{inx} dx}^{2\pi\,\bar{c}_n \text{ by def.}}$$

$$- \sum_n \bar{c}_n \underbrace{\int_{-\pi}^{\pi} f(x) e^{-inx} dx}_{2\pi\, c_n \text{ by def}} + \sum_{n,m} c_n \bar{c}_m \underbrace{\int_{-\pi}^{\pi} e^{inx} e^{-imx} dx}_{=0 \text{ if } n \neq m \text{ by orthogonality}} =$$

$$= \int_{-\pi}^{\pi} |f(x)|^2 dx - 2\pi \sum_n |c_n|^2 - 2\pi \sum_n |c_n|^2 + 2\pi \sum_n |c_n|^2$$

$$= \int_{-\pi}^{\pi} |f(x)|^2 dx - 2\pi \sum_{n=-N}^{N} |c_n|^2$$

Hence, $\sum_{n=-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$ for any $N \in \mathbb{N}$.

Taking $N \to \infty$ gives $\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$ (-)

P **Lemma (Riemann-Lebesgue)**

Under the above assumptions, one has

$\lim_{|n| \to \infty} c_n = 0$ and $\lim_{n \to \infty} a_n = 0$ and $\lim_{n \to \infty} b_n = 0$.

Proof:

Bessel ineq. gives us

$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx \overset{f \in L^2}{<} \infty$, hence the series $\sum_{n=-\infty}^{\infty} |c_n|^2$ converges

hence (*) $|c_n|^2$ must tend to zero, hence $|c_n|$ must tend to zero and so do a_n and b_n (by def).

[(*) $c_N := S_N - S_{N-1}$, partial sums, $\lim_{N \to \infty} c_N = c - c = 0$]

Rem: If a_n, b_n, or $c_n \underset{n \to \infty}{\not\to} 0$, then the Fourier series of f cannot converge!

Ex: (see above expl.)

We have considered $f(x) = x$ for $0 < x \leq 2\pi$.

We compute $\int_{-\pi}^{\pi} |f(x)|^2 dx \underset{f \text{ } 2\pi\text{-periodic}}{=} \int_0^{2\pi} |f(x)|^2 dx = \frac{x^3}{3}\Big|_0^{2\pi} = \frac{8\pi^3}{3} < \infty$

Hence, by Riemann-Lebesgue, one has $a_n, b_n \underset{n \to \infty}{\longrightarrow} 0$.

This is indeed the case as $a_n = 0$ and $b_n = -\frac{2}{n}$ (see ~~above~~ notes)

6) Convergence of Fourier series:

Recall: A series $\sum_{n=1}^{\infty} x_n$ converges to some $x \in \mathbb{R}$ if its partial sums $S_N := \sum_{n=1}^{N} x_n$ converges to x, that is $\lim_{N\to\infty} S_N = x$.

Th: (Dirichlet theorem)

Let f be 2π-periodic with f and f' piecewise continuous on $[0,2\pi]$. Set $S_N^f(x) := \sum_{n=-N}^{N} c_n e^{inx}$, where c_n are the Fourier coeff. of f. Then,

$$\lim_{N\to\infty} S_N^f(x_0) = \frac{f(x_0-)+f(x_0+)}{2} \qquad \forall x_0 \in [0,2\pi] \text{ or } \mathbb{R}$$

If f is continuous at x_0, then $\lim_{N\to\infty} S_N^f(x_0) = f(x_0)$!!

Rem: $f(x_0-) := \lim_{x\to x_0^-} f(x)$, $f(x_0^+) = \lim_{x\to x_0^+} f(x)$

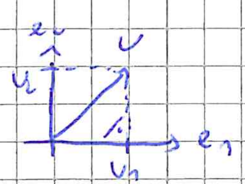
Th (Parseval identity)

For $f \in L^2(-\pi,\pi)$ 2π-periodic with f, f' pwc, one has

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Or, see compendium, $\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}\left(|a_n|^2+|b_n|^2\right)$

Rem. Parseval is Pythagoras for Fourier-series!



$$v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n \Rightarrow \|v\|^2 = \langle v,v\rangle = \sum_{j=1}^{n} v_j^2$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \to 2\pi \cancel{\sum_{n=-\infty}^{\infty}} \|f\|_{L^2}^2 = \int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Parseval

Parseval helps to compute (some) series:

Ex: We know:

(see above expl) $f(x) = x = \sum_{n=1}^{\infty} \underbrace{\frac{2(-1)^{n+1}}{n}}_{b_n} \sin(nx)$