**Th:** (superposition principle)

If $u_1, u_2$ satisfy linear PDEs $L(u_1) = F_1$ and $L(u_2) = F_2$.

Then $u = c_1 u_1 + c_2 u_2$, $\forall c_1, c_2 \in \mathbb{R}$, satisfy the prob.

$$\underbrace{L(u) = c_1 F_1 + c_2 F_2}_{\text{can be more complicated}}$$

[read from right to left]

**Ex:** Consider Laplace eq. $L(u) = u_{xx}(x,y) + u_{yy}(x,y) = 0$.

Since $u_1(x,y) = x^2 - y^2$ and $u_2(x,y) = e^x \cos(y)$ are sol. to $L(u) = 0$, then any linear combination $u = c_1 u_1 + c_2 u_2$, $\forall c_1, c_2 \in \mathbb{R}$ satisfy $L(u) = 0$!

Verification: $u_{xx}(x,y) = c_1 \cdot 2 + c_2 (e^x \cos(y))$

$u_{yy}(x,y) = c_1(-2) + c_2(-e^x \cos(y))$ $\Rightarrow u_{xx} + u_{yy} = 0$!

**Application:**

Want to solve (*) $\begin{cases} L(u) = F \neq 0 \\ B(u) = f \end{cases}$ (linear $B()$)

One needs "only" to solve the homog. problem $\begin{cases} L(u) = 0 \\ B(u) = 0 \end{cases}$

and find a particular sol. to $\begin{cases} L(v) = F \\ B(v) = f \end{cases}$ to get the

general sol to $\begin{cases} L(u) = F \\ B(u) = f \end{cases}$

Indeed: $\forall u$ sol. to (*), $v$ part. sol, then $w = u - v$ solves homog. prob. Hence $u = \underbrace{w}_{\text{sol. hom.}} + \underbrace{v}_{\text{sol. particular}}$ (same as for ODE)

## 2) Separation of variables:

**Idea:** To solve a linear hom. PDE with homo. BC, one makes the ansatz $u(x,t) = X(x) \cdot T(t)$ and solve "easier" prob. for $X$ and $T$.

Ex: (Heat eq)

Let $k \in \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}$, $L > 0$, $T > 0$ and consider

$$\text{(H)} \begin{cases} u_t - k u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & 0 < x < L \end{cases}$$

(i) Insert $u(x,t) = X(x) \cdot T(t)$ into PDE:

$X(x) \cdot \dot{T}(t) - k X''(x) \cdot T(t) = 0$ or $X(x) \cdot \dot{T}(t) = k \cdot X''(x) \cdot T(t)$ or

$$\underbrace{\frac{\dot{T}(t)}{k \cdot T(t)}}_{\text{only "}t\text{"}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{only "}x\text{"}} \overset{!}{=} \lambda \quad \text{constant!!}$$

indep. variables

(ii) Get 2 "simple" ODEs: $\dot{T}(t) = \lambda \cdot k \cdot T(t)$

$X''(x) - \lambda X(x) = 0$

(iii) Solve ODE: $T(t) = C \cdot e^{\lambda k t}$, with some constant $C$.

$X(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$, where $\omega = \sqrt{-\lambda}$

(iv) Use BC $u(0,t) = 0 = X(0) \cdot T(t) = (c_1 + c_2 \cdot 0) \cdot T(t)$

$c_1 \overset{!}{=} 0!$ $\quad T(t) \equiv 0 \Rightarrow u(x,t) = 0$ not interesting

$$u(L,t) = 0 = X(L) \cdot T(t) = (\underbrace{c_2 \sin(\omega L)}_{\overset{!}{=} 0}) \cdot \underbrace{T(t)}_{\neq 0}$$

($c_2 \neq 0$, else $X \equiv 0$ and $u \equiv 0$)

$\Leftrightarrow \omega L = n\pi$ for some integers $n$ or $\omega = \frac{n\pi}{L}$ or $\lambda = -\left(\frac{n\pi}{L}\right)^2 \quad \forall n \in \mathbb{N}$ $(n \neq 0)$

(v) For any $n = 1, 2, 3, 4, \ldots$: we have obtained solutions

$$u_n(x,t) = X_n(x) \cdot T_n(t) = \sin\left(\frac{n\pi x}{L}\right) \cdot \exp\left(-\left(\frac{n\pi}{L}\right)^2 k \cdot t\right)$$

By superposition $\quad u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-\left(\frac{n\pi}{L}\right)^2 k t\right) \sin\left(\frac{n\pi x}{L}\right)$

assuming convergence

(vi) We still have to use IC $u(x,0)=f(x)$ !!

$$f(x)=u(x,0)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$ which is a Fourier series !!

Hence $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ for $n=1,2,\dots$

$\leadsto\leadsto$ $u(x,t)=\sum_{n=1}^{\infty} b_n \exp\left(-\left(\frac{n\pi}{L}\right)^2 kt\right)\sin\left(\frac{n\pi x}{L}\right)$ is sol. to (H) !!

See lecture notes for an application with $k=\frac{1}{10}$, $L=1$, $f(x)=x(1-x)$

Rem. Above works also for other hom. BC: $u_x(0,t)=u_x(L,t)=0$
and for other PDE (lin. homog.)

Ex. (wave eq.)

Let $c>0$, $L>0$, and $f,g: \mathbb{R}\to\mathbb{R}$, consider

$$(W)\begin{cases} u_{tt}(x,t)-c^2u_{xx}(x,t)=0 & , 0<x<L, 0<t \\ u(0,t)=u(L,t)=0 & , t>0 \\ u(x,0)=f(x), \ u_t(x,0)=g(x) & , 0<x<L \end{cases}$$

(i) Insert $u(x,t)=X(x)\cdot T(t)$ into DE to get

$$\ddot{T}X-c^2TX''=0 \quad \text{or} \quad \underbrace{\frac{\ddot{T}}{c^2T}}_{\text{only "t"}}=\underbrace{\frac{X''}{X}}_{\text{only "x"}}=-\mu^2$$

(ii) Get 2 simple ODEs: $X''+\mu^2X=0$

$$\ddot{T}+c^2\mu^2T=0$$

(iii) Solve BVP $\begin{cases} X''+\mu^2X=0 \\ X(0)=X(L)=0\end{cases}$ $\leadsto$ $X(x)=A\cos(\mu x)+B\sin(\mu x)$

Use BC to find $A,B,\mu$ $\leadsto$ $X(0)=0=A+0 \Rightarrow A=0$

$X(L)=0=B\sin(\mu L) \underset{B\neq 0}{\Rightarrow} \sin(\mu L)=0$

$\Rightarrow \mu L=n\pi$ for $n=1,2,3,\dots$, or $\mu=\frac{n\pi}{L}$

Gives sol. $X_n(x)=B_n\sin\left(\frac{n\pi}{L}x\right)$ for $n=1,2,3\dots$

(iv) Similarly, get $T_n(t) = C_n \cos\left(\frac{cn\pi}{L} t\right) + D_n \sin\left(\frac{cn\pi}{L} t\right)$ for $n = 1, 2, \ldots$

(v) By superposition, this gives us the sol. to (W):

$$u(x,t) = \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{c \cdot n \cdot \pi}{L} \cdot t\right) + b_n \sin\left(\frac{c \cdot n \cdot \pi}{L} \cdot t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

and its derivative

$$u_t(x,t) = \sum_{n=1}^{\infty} \left( \left(-\frac{cn\pi}{L}\right) a_n \sin\left(\frac{cn\pi}{L} \cdot t\right) + \frac{c \cdot n\pi}{L} b_n \cos\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

(vi) Use IC to find $a_n$ and $b_n$!

$$u(x,0) \overset{IC}{=} f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L} x\right) \overset{\text{Fourier!}}{\Rightarrow} a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx \quad ;-)$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \left(\frac{cn\pi}{L}\right) b_n \sin\left(\frac{n\pi x}{L}\right) \overset{FS}{\Rightarrow} \frac{cn\pi}{L} b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

↳ With above $a_n, b_n$, one gets $u(x,t) = \sum_{n=1}^{\infty} \ldots$ sol. to (W)!!

## 3) Inhomogeneous equations:

What can we do if the PDE or the BC are inhomogeneous?

Idea: Superposition: split the prob into "easier" parts!

Ex: Solve the inhomogeneous heat eq.

$$(H) \begin{cases} u_t - u_{xx} = \boxed{8} \\ u(0,t) = \boxed{1}, \; u(1,t) = \boxed{3} \\ u(x,0) = f(x) \end{cases}$$

where $f(x)$ is given.

To find a sol. to (H), split $u(x,t) = v(x,t) + s(x)$, where

$v(x,t) \rightsquigarrow$ transient sol. (verify IC and vanish as $t \to \infty$)

$s(x) \to$ steady state sol. (does not depend on time, hence $s_t \equiv 0$)

Insert this ansatz into DE to get:

$$v_t - v_{xx} - s'' \overset{!}{=} 8.$$

Set $-s''(x) \overset{!}{=} 8$ so that $v_t(x,t) - v_{xx}(x,t) = 0$

BVP ;-) homog. heat ;-)