(Parseval) $\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2+|b_n|^2)$

Hence, $\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx}_{\frac{2\pi^2}{3}} = \sum_{n=1}^{\infty}\frac{4}{n^2}$

Or: $\sum_{n=1}^{\infty}\frac{1}{n^2} = 1+\frac{1}{4}+\frac{1}{9}+\dots = \frac{\pi^2}{6}$, Basel problem solved :-)
(1650 - 1735 Euler)

7) Functions of arbitrary period:

What can we say if f is not 2π-periodic, but say $2L$-periodic?

Consider $f(x)$ for $-L\le x\le L$ with period $p=2L$.

Idea: Change of variable: $[-L,L] \longrightarrow [-\pi,\pi]$

$x \longmapsto t=\frac{\pi}{L}x$

Define "new" fct $g(t) := f\left(\frac{L}{\pi}t\right) = f(x)$.

$t=\frac{\pi}{L}x$ or $x=\frac{Lt}{\pi}$

This fct has period $p=2\pi$:

$g(t+2\pi) \underset{\text{Def } g}{=} f\left(\frac{L}{\pi}(t+2\pi)\right) = f\left(\frac{Lt}{\pi}+2L\right) \underset{f\ 2L\text{-periodic}}{=} f\left(\frac{Lt}{\pi}\right) \underset{\text{Def } g}{=} g(t)$:-)

The FS of $g(t)$ is then given by $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nt)+b_n\sin(nt))$

or in terms of variable x:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi}{L}x\right)+b_n\sin\left(\frac{n\pi}{L}x\right)\right)$$

which is the FS of $f(x)$ (since $g(t)=f(x)$)

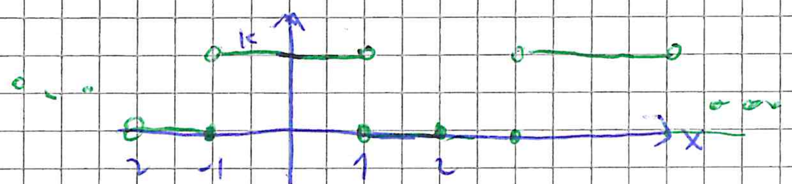
The Fourier coeff are computed using

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}g(t)\cos(nt)\,dt \underset{\substack{t=\frac{\pi}{L}x,\ dt=\frac{\pi}{L}dx\\ g(t)=f(x)}}{=} \frac{1}{\pi}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi}{L}x\right)\frac{\pi}{L}dx.$

$$\boxed{\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad n=0,1,2,\dots \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \qquad n=1,2,\dots \end{aligned}}$$

<u>Ex:</u> Let $k \in \mathbb{R}$. Find the FS of

$$f(x) = \begin{cases} 0 & -2 < x \leq -1 \\ k & -1 < x < 1 \\ 0 & 1 \leq x \leq 2 \end{cases} \quad \text{extended periodically}$$



$p = 4 = 2 \cdot 2 \Rightarrow L = 2$

<u>Obs:</u> f is even $\Rightarrow b_n = 0$ for $n = 1,2,3,\dots$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \underset{f \text{ even}}{=} \frac{2}{L}\int_0^L f(x)\,dx \underset{L=2}{=} 1 \cdot \int_0^2 f(x)\,dx \underset{\text{Def } f(x)}{=} k$$

$$a_n = \dots = \frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = \frac{k}{2} + \frac{2k}{\pi}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos\left(\frac{n\pi x}{2}\right)$$

8) Sine and cosine series

What can we say if f is <u>not</u> periodic?

Consider $f(x)$ for $0 \leq x \leq L$:

(i) Extend f in a simple way to $[-L, L]$

(ii) Extend the result of (i) periodically. (done above :-))

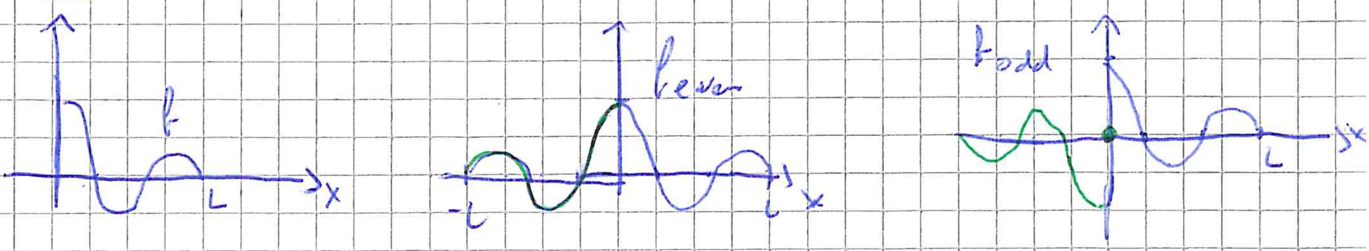
There are 2 standard ways to do (i): make f even <u>or</u> odd.

<u>Def:</u>
- The <u>even extension f_{even} of f</u> to $[-L, L]$ is defined by

  $f_{even}(-x) = f(x)$ for $x \in [0, L]$

- The <u>odd extension f_{odd}</u> is defined by $f_{odd}(-x) = -f(x)$ for $x \in (0, L]$ and $f(0) = 0$.

Ex:



Rem: f_{even} is even $\Rightarrow$ Fourier coeff $b_n = 0 \Rightarrow$ FS has only cosines

f_{odd} is odd $\Rightarrow$ Fourier coeff $a_n = 0 \Rightarrow$ FS has only sines

Def: For f integrable on $[0, L]$, the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \text{ with } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

is called the Fourier cosine series of f.

The series $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$, where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

is called the Fourier sine series ~~of~~ of f.

Ex: See compendium

9) Derivatives and integrals of Fourier series:

Prob: How can one compute the FS of f'?

Ex. The FS of $f(x) = x$, for $-L \le x \le L$, $2L$-periodic reads

$$f(x) = x = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{L}x\right)$$

Try to differentiate term by term and get:

$$x' = 1 = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cos\left(\frac{n\pi}{L}x\right)\cdot\frac{n\pi}{L} = \underbrace{2\cdot\sum_{n=1}^{\infty}(-1)^{n+1}\cos\left(\frac{n\pi}{L}x\right)}_{\text{does not converge !!}}$$

$[(-1)^{n+1}\cos(\dots) \not\to 0]$

$\hookrightarrow$ be carefull!!!

Th: If $f: \mathbb{R} \to \mathbb{R}$ continuous and $2L$-periodic, if f' and f'' pwc on $[-L, L]$. Then, the Fourier coeff of f' are given by

$a_n' = n b_n$, $b_n' = -n a_n$, $c_n' = -i n c_n$

Obs: $f(x) = \sum_n c_n e^{inx} \rightsquigarrow f'(x) = \sum_n i n c_n e^{inx}$ $\forall x$, where $f'(x)$ $\exists$.

Th: Suppose f 2π-periodic and pwc with Fourier coeff a_n, b_n, c_n.
Define $F(x) = \int_0^x f(y)\,dy$. If $c_0 = \frac{1}{2}a_0 = 0$, then $\forall x \in \mathbb{R}$, one has

$$F(x) = C_0' + \sum_{n \neq 0} \frac{c_n}{in} e^{inx} = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin(nx) - \frac{b_n}{n} \cos(nx) \right),$$

integrate term by term

where $C_0' = \frac{1}{2} A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)\,dx$.

If $c_0 \neq 0$ (small), the sum of the series on the right is $F(x) - c_0 x$ (small).

Rem: • f periodic but F may not! $f(x) = 1$ periodic $\rightsquigarrow F(x) = x$ not!
• integral of constants is not periodic, hence the above condition $c_0 = 0$.

Chapter X: Partial differential equations

Goal: Present a tool to find exact sol. to particular PDE (using compendium)

1) Linearity and superpositions

Recall: Heat eq. $u_t(x,t) - u_{xx}(x,t) = f(x)$, wave eq. $u_{tt}(x,t) - c(x) u_{xx}(x,t) = f(x)$

We shall more generally consider linear PDE

$$\boxed{L(u) = F},$$

with a given linear operator L and given fct $F = F(x)$.

$(L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2) \quad \forall c_1, c_2 \in \mathbb{R}, u_1, u_2 \text{ fct})$

Ex: Heat $L(u) = u_t - u_{xx}$, Wave $L(u) = u_{tt} - c(x) u_{xx}$

General: $L(u) = a(x) u + b(x) \frac{\partial u}{\partial x} + c(x) \frac{\partial^2 u}{\partial x^2}$

Rem: $F \equiv 0 \Rightarrow$ one has a linear homogeneous PDE

else it is inhomogeneous