Recall: • $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$ (FS)

2π-periodic

$$\sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx \quad \text{for } n = 0, 1, 2, \dots$$

• $f(x) = \underbrace{\sum_{n=-\infty}^{\infty} c_n e^{inx}}_{\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{inx}}$ for f 2π-periodic, continuous in x and f' pwc.

Parseval: For $f \in L^2(-\pi, \pi)$ 2π-periodic with

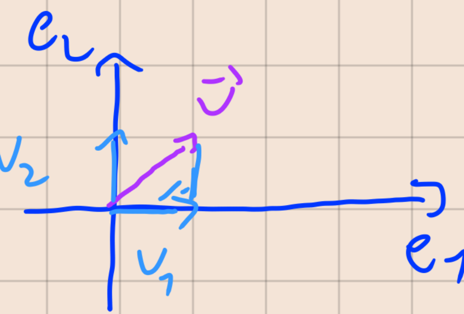
f, f' pwc. Then,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{|a_0|^2}{2} + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$$

lecture notes

Rem: "Parseval is Pythagoras in Fourier"



$\mathbb{R}^2$ $\quad \vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2$

Pythag. $\|\vec{v}\|^2 = v_1^2 + v_2^2 = \sum_{n=1}^{2} v_n^2$

L^2 $\quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

Parseval $\quad \frac{1}{2\pi} \|f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$

Why should we use Parseval?

Ex: (Compute series!)

From a previous example, we consider

$$f(x) = x = \sum_{n=1}^{\infty} \underbrace{\frac{2(-1)^{n+1}}{n}}_{b_n} \sin(nx) \quad \text{(FS)}$$

From Parseval, we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} |\underbrace{f(x)}_{x}|^2 \, dx = \frac{|\overbrace{a_0}|^2}{2}\Big|_{a_0 = 0} + \sum_{n=1}^{\infty} \left(|\underbrace{a_n}_{0}|^2 + |b_n|^2 \right)$$

This gives us

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \sum_{n=1}^{\infty} \frac{4}{n^2} \qquad \text{or}$$

$$\frac{1}{\pi}\left.\frac{x^3}{3}\right|_{-\pi}^{\pi} = 4\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad \text{or}$$

$$\frac{2\pi^3}{3\cdot\pi\cdot 4} = \sum_{n=1}^{\infty}\frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Basel problem (stated 1650
Euler solved 1735)

7) Functions of arbitrary periods!

Question: Can we do a FS when f is not 2π-periodic, f.ex. if f is $2L$-periodic?

Consider f to be $2L$-periodic and defined on $[-L, L]$.

Idea: Change variable

$$[-L, L] \longrightarrow [-\pi, \pi]$$

$$x \longmapsto t = \frac{\pi x}{L}$$

And consider the "new" function

$$g(t) = f\left(\frac{Lt}{\pi}\right) \overset{\uparrow}{=} f(x).$$

$$x = \frac{Lt}{\pi}$$

We show that g is 2π-periodic:

Def g

$$g(t + 2\pi) \overset{\downarrow}{=} f\left(\frac{L}{\pi}(t + 2\pi)\right) = f\left(\frac{Lt}{\pi} + 2L\right) =$$

f $2L$-periodic

Def. of g

$$\overset{\downarrow}{=} f\left(\frac{Lt}{\pi}\right) \overline{\overline{\qquad\qquad\qquad\qquad}} g(t)$$

Thus, the Fourier series of g reads

$$g(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right),$$

$\shortparallel$

$f(x)$

where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(nt) \, dt$

for $n = 0, 1, 2, \dots$

This gives us the FS of f:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L} \right) + b_n \sin\left(\frac{n\pi x}{L} \right) \right)$$

$t = \frac{\pi x}{L}$

$dt = \frac{\pi}{L} dx$

where

$$a_n = \frac{1}{\pi} \int_{-L}^{L} f(x) \cos\left(n \frac{\pi x}{L} \right) dx \cdot \frac{\pi}{L}$$

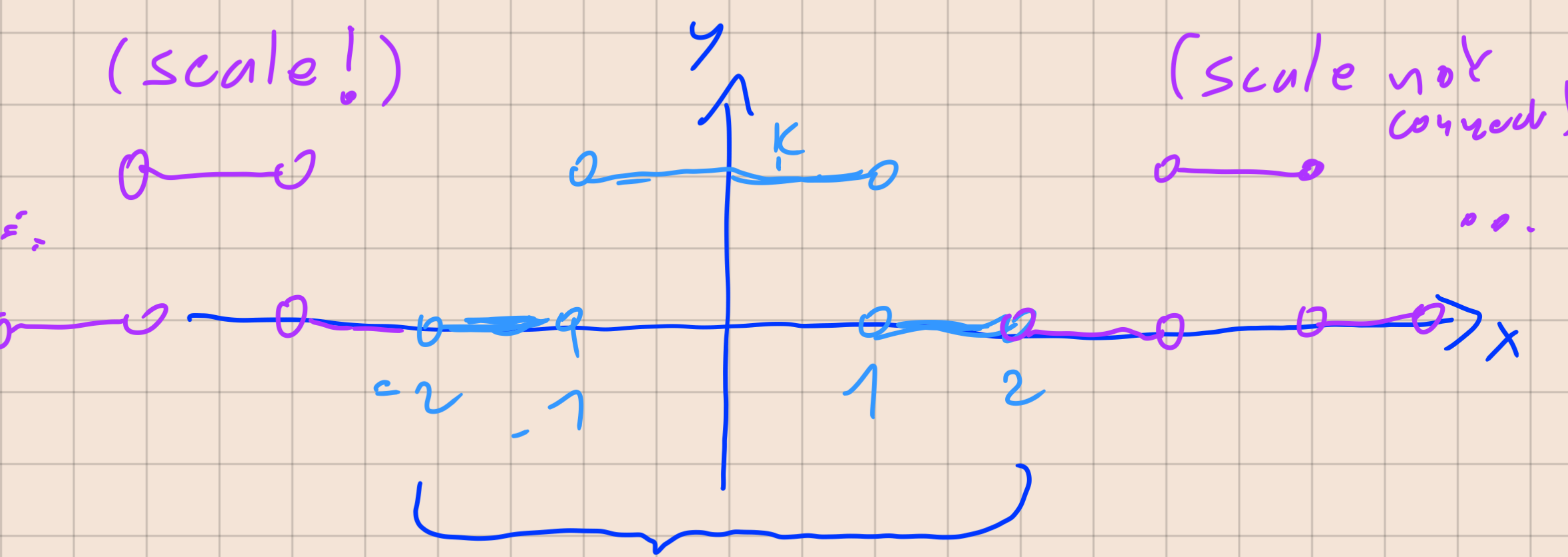
$$= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L} \right) dx, \text{ for } n = 0, 1, \dots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n=1,2,\dots$$

<u>Ex:</u> Let $k\in\mathbb{R}$. Find the FS of

$$f(x)=\begin{cases}0 & -2<x<-1\\ k & -1<x<1\\ 0 & 1<x<2\end{cases}$$

and <u>extend</u> periodically:



f is 4-periodic with $4 = 2\cdot \underbrace{2}_{= L}$

coeff. in front sin

Since f is even $\Rightarrow b_n = 0$ for $n=1,2,\dots$

and $a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}} dx =$

$$= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \overset{\text{Def } f,\ L=2}{=}$$

$$= \frac{2}{2}\cdot\int_0^2 f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

For a_0, we get:

$$a_0 = \int_0^2 f(x)\,dx \underset{\text{Def } f}{=} \int_0^1 k\,dx + \int_1^2 0\,dx =$$

$$= k$$

For $n \neq 0$, we get $a_n = \ldots = \frac{2k}{n\pi}\sin\left(\frac{n\pi}{2}\right)$

And finally, the FS of f reads

$$f(x) \sim \underbrace{\frac{k}{2}}_{\left(\frac{a_0}{2}\right)} + \sum_{n=1}^{\infty} \underbrace{\frac{2k}{n\pi} \sin\left(\frac{n\pi}{2}\right)}_{a_n} \cdot \cos\left(\frac{n\pi x}{2}\right) \quad //$$

8) Sine-cosine Fourier series:

Question: What can we do if f is not periodic?

Consider f defined for $x \in [0, L]$

Idea: (i) Extend f in a simple way to $[-L, L]$

(ii) Extend (i) periodically and use previous section

In order to address (i), we consider

2 standard ways:

Def: • The <u>even extension f_{even}</u> of f to the interval $[-L, L]$ is defined by

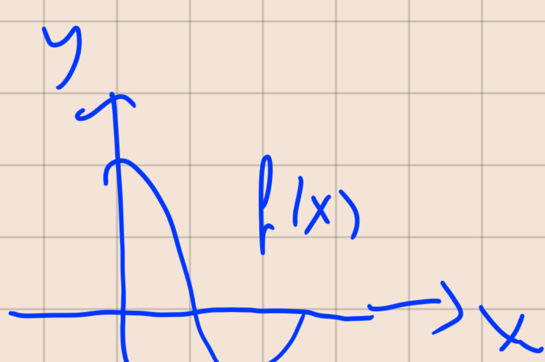
$$f_{even}(-x) = f(x) \text{ for } x \in [0, L]$$

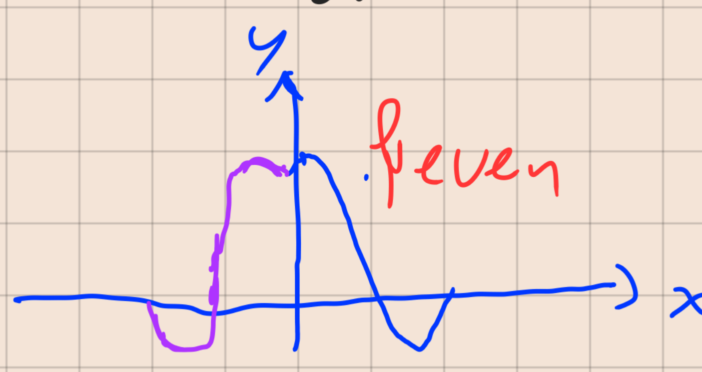
(and $f_{even}(x) = f(x)$ for $x \in [0, L]$)

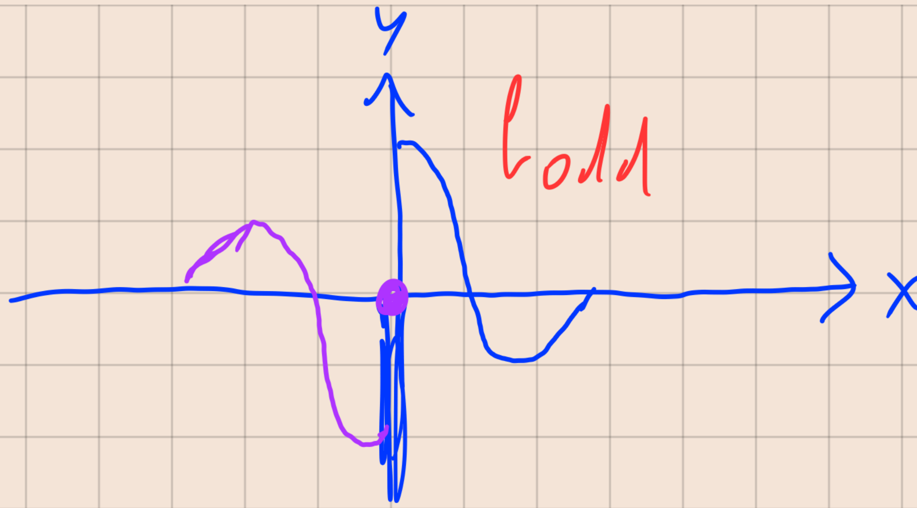
• The <u>odd extension f_{odd}</u> of f to $[-L, L]$ is defined by

$$f_{odd}(-x) = -f(x) \text{ for } x \in (0, L]$$

and $f_{odd}(0) = 0$







Rem: • f_{even} is even $\Rightarrow$ Fourier coeff $b_n = 0$

$\Rightarrow$ FS has only cosine terms!

• f_{odd} is odd $\Rightarrow$ Fourier coeff $a_n = 0$

$\Rightarrow$ FS has only sine terms!

Def: For an integrable function f on $[0, L]$, the series

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{with}$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

is called the Fourier cosine series of f.

The series $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

with $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

is called the Fourier sine series of f.

$$f(x) \sim \frac{a_0}{2}\cdot\overbrace{\cos(0x)}^{1} + a_1\cos(1x) + a_2\cos(2x) + \ldots$$
$$+ b_1\sin(1x) + b_2\sin(2x) + \ldots$$

9) Derivatives and integrals of Fourier series:

Rem: All assumptions on $f, f', \ldots$ are in the summary.

We consider $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

with Fourier coeff. c_n.

<u>Question</u> What is the FS of $f'(x)$?

<u>Answer</u>: $f'(x) = \sum_{n=-\infty}^{\infty} \underbrace{inc_n}_{\text{Fourier coeff } f'} e^{inx}$

<u>Question</u>: What is the FS of

$$F(x) = \int_0^x f(y)\,dy$$

<u>Answer</u>: If $c_0 = 0$, then one has

$$F(x) = C'_0 + \sum_{n \neq 0} C'_n e^{inx}, \text{ where}$$

$$C'_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)\,dx \quad \text{and}$$

$$C_n' = \frac{c_n}{in} \qquad \left(\int e^{inx}\,dx = \frac{e^{inx}}{in} \right)$$