Recall: • FS of f: $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$

• Fourier sine series $f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

defined on $[0, L]$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

• $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \Rightarrow f'(x) \sim \sum_{n=-\infty}^{\infty} in c_n e^{inx}$

$\Rightarrow f''(x) \sim \sum_{n=-\infty}^{\infty} (in)^2 c_n e^{inx}$, etc.

• $F(x) = \int_0^x f(y)dy \underset{c_0=0}{\sim} C_0 + \sum_{n\neq 0} \frac{c_n}{in} e^{inx}$

Chapter X: Partial differential equations

Goal: Present and use a tool to find solutions to particular PDE.

1) Linearity and superposition:

Recall: Heat eq. $u_t(x,t) - u_{xx}(x,t) = g(x)$, $u_t = \frac{\partial u}{\partial t}$

Wave eq. $u_{tt}(x,t) - c(x)\,u_{xx}(x,t) = g(x)$

etc.

Def: We shall consider linear PDE of the form $L(u) = F$, where $F = F(x)$, and L is a linear operator.

I.e. $L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$

for any $c_1, c_2 \in \mathbb{R}$ and functions u_1, u_2.

Ex: • For the above heat eq, one has

$L(u) \equiv u_t - u_{xx}$ and $F(x) = g(x)$.

• For the above wave eq, one has

$L(u) = u_{tt} - c(x)\,u_{xx}$, $F(x) = g(x)$.

• $L(u) = u_t - u_x^2$ is not a linear operator!

Def: The linear PDE $L(u)=F$ is called **homogeneous** if $F \equiv 0$, and **inhomogeneous** else.

Def: We shall consider **linear BC** of the form $B(u)=f$, where $f=f(x)$, and B is a linear operator.

Ex: Dirichlet BC in 1d: $u(0)=u(1)=0$ is of the form $B(u)=f$, where $f=0$ and $B(u)=u$ on the boundary.

Th: (Superposition principle)

If u_1 and u_2 satisfy the linear

PDEs $L(u_1) = F_1$ and $L(u_2) = F_2$, then

$u \stackrel{\text{Def}}{=} c_1 u_1 + c_2 u_2$, where $c_1, c_2 \in \mathbb{R}$,

satisfies the prob

$$L(u) = c_1 F_1 + c_2 F_2$$

<u>Proof:</u>

$$L(u) \underset{\uparrow}{\equiv} L(c_1 u_1 + c_2 u_2) \underset{\uparrow}{=} c_1 L(u_1) + c_2 L(u_2) =$$

Def u linear PDE

$$\underset{\uparrow}{=} c_1 F_1 + c_2 F_2$$

Def u_1, u_2

:-)

∎

<u>Ex:</u> Consider Laplace eq.

$$L(u) = u_{xx}(x,y) + u_{yy}(x,y) = 0$$

Since $u_1(x,y) = x^2 - y^2$ and

$u_2(x,y) = e^x \cos(y)$ are sol. to $L(u) = 0$

then the superposition principle tells us that $u = c_1 u_1 + c_2 u_2$ is a sol. to $L(u) = 0$!!

Indeed: $L(u) \underset{\text{Def } L}{=} u_{xx} + u_{yy} \underset{\text{Def } u}{=} (c_1 u_1 + c_2 u_2)_{xx}$

$+ (c_1 u_1 + c_2 u_2)_{yy} \underset{\text{linearity}}{=} c_1 (u_1)_{xx} + c_2 (u_2)_{xx}$

$+ c_1 (u_1)_{yy} + c_2 (u_2)_{yy} \underset{\text{Def } u_1, u_2}{=}$

$= 2c_1 + e^x \cos(y) c_2 + c_1(-2)$

$+ c_2(-e^x \cos(y)) = 0$!!

$y'' - y + y' = 0$

$y_1, y_2 \Rightarrow c_1 y_1 + c_2 y_2$

Application: Superposition tells us:

In order to solve inhomogeneous lin. PDE

$$(P)\begin{cases} L(u) = F \\ B(u) = f \end{cases}$$

One "just" needs

to find the general sol. to hom. prob.

$$(H)\begin{cases} L(u) = 0 \\ B(u) = 0 \end{cases}$$

and one particular sol. to (P).

Suppose that u is any sol. to (P).

Consider $w := u - u_p$, where u_p is a particular sol. Then, $L(w) = L(u - u_p) = L(u) - L(u_p) =$

$$= F - F = 0!$$

$$B(w) = 0$$

This means that w solves (H). Or,

$$u = \underset{\substack{\downarrow \\ \text{sol. (H)}}}{w} + \underset{\substack{\downarrow \\ \text{particular sol.}}}{u_p}$$

⌈ Same as for ODE: $y'' - 2y' + y = \sin(x)$

Sol. $y(x) = \underset{\substack{\downarrow \\ \text{sol. } y''-2y'+y}}{y_H(x)} + \underset{\substack{\downarrow \\ \text{particular sol.}}}{y_p(x)}$ ⌋

2) Separation of variables:

Goal: Find sol. to lin. hom. PDE

$$(H)\begin{cases} L(u) = 0 \\ B(u) = 0 \end{cases}$$

Idea: Write $u(x,t) = X(x) \cdot T(t)$ and solve 2 ODEs for X and T.

Ex: (Heat eq.)

Let $k \in \mathbb{R}$, $L>0$, $T>0$, $f: \mathbb{R} \to \mathbb{R}$ given. Consider

$$\begin{cases} u_t(x,t) - k \cdot u_{xx}(x,t) = 0 & \text{DE} \\ u(0,t) = 0 = u(L,t) & \text{BC (hom. Dirichlet)} \end{cases}$$

$\begin{cases} u(x,0) = f(x) \end{cases}$ IC

Idea: $u(x,t) = X(x) \cdot T(t)$ insert in DE:

$$u_t - k\, u_{xx} = 0 \iff X(x) \cdot T'(t) - k \cdot X''(x) \cdot T(t) = 0$$

$$\iff X(x) \cdot T'(t) = k \cdot X''(x) \cdot T(t) \iff \underbrace{\frac{T'(t)}{k \cdot T(t)}}_{\text{only depends on } t!} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{only dep. on } x!} \overset{!}{=} \lambda$$

where λ is some constant!!

The above gives us 2 ODEs:

$$\begin{cases} T'(t) = k \cdot T(t)\, \lambda \\ X''(x) = \lambda X(x) \end{cases}$$

The first ODE $T'(t) = k \cdot \lambda \cdot T(t)$ has solution $T(t) = \exp(k \cdot \lambda \cdot t)$!

The second ODE, $X''(x) - \lambda X(x) = 0$, has solution $X(x) = A \cdot \cos(\sqrt{-\lambda}\,x) + B\sin(\sqrt{-\lambda}\,x)$

with some constants A and B,

$$\left[\ A\cos(\sqrt{-\lambda}\,x) \rightsquigarrow A\cdot\sqrt{-\lambda}\,\sin(\sqrt{-\lambda}\,x) \rightsquigarrow -A(-\lambda)\cos(\sqrt{-\lambda}\,x)\ \right]$$

Using BC, we shall find A, B, λ:

We must have $u(0,t) = 0 \Leftrightarrow X(0)\cdot T(t) = 0$,

So either $T(t) = 0$ or $X(0) = 0$.

$\hookrightarrow$ This implies $u(x,t) = X(x)T(t) = 0$

not interesting!

The condition $X(0) = 0$ gives us

$$X(0) \underset{\text{Def } X}{=} A\cos(0) + B\sin(0) = A + 0 = A = 0$$

Which implies that $A = 0$!

We next use the other BC:

$u(L,t)=0 \Leftrightarrow X(L)\cdot T(t)=0$

($T\equiv 0$ not interesting)

$\Leftrightarrow X(L)=0$ or $X(L)=B\cdot\sin(\sqrt{-\lambda}\cdot L)=0$

($B\equiv 0$, not interesting: $\rightarrow X(x)=0 \Rightarrow u\equiv 0$)

$\Rightarrow \sin(\sqrt{-\lambda}\cdot L)=0 \Rightarrow \sqrt{-\lambda}\cdot L=n\cdot\pi$

for $n=1,2,3,\ldots$

$\Rightarrow \lambda=-\left(\frac{n\pi}{L}\right)^2$ for $n=1,2,3,\ldots$

Finally, we get the solutions

$X_n(x)=B_n\cdot\sin\left(\frac{n\pi}{L}\cdot x\right)$ for $n=1,2,3\ldots$!!

By superposition and def. of u, we know that

$$u(x,t) = \sum_{n=1}^{\infty} B_n \underbrace{\exp\left(-k\left(\frac{n\pi}{L}\right)^2 \cdot t\right)}_{T(t)} \cdot \underbrace{\sin\left(\frac{n\pi}{L}x\right)}_{X_n(x)}$$

is sol. to DE + BC !!