Recall: • Fourier sine series

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \qquad n = 1,2,\ldots$$

• Linear PDE $\quad L(u) = F$

• $\begin{cases} u_t(x,t) - k\,u_{xx}(x,t) = 0 & \text{(DE)} \\ u(0,t) = u(L,t) = 0 & \text{(BC)} \\ u(x,0) = f(x) & \text{(IC)} \end{cases}$ hom. heat eq.

(i) Ansatz $\quad u(x,t) = X(x)\cdot T(t)$ $\qquad \dfrac{T'(t)}{kT(t)} = \dfrac{X''(x)}{X(x)} = \lambda$

(ii) ODE $\quad T'(t) = \lambda k T(t)$

BVP $\quad X''(x) - \lambda X(x) = 0$
$\quad X(0) = X(L) = 0$

(iii) $\quad X_n(x) = B_n \sin\left(\frac{n\pi}{L}x\right) \qquad n: 1,2,\ldots \qquad \lambda_n = -\left(\frac{n\pi}{L}\right)^2$ $\;(-\mu^2)$

$\quad T_n(t) = \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right) \qquad n = 1,2,\ldots$

$\overset{(L)}{\Longrightarrow}$ $\quad u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right)\sin\left(\frac{n\pi}{L}x\right)$

Superposition

We still need to use IC! This will help us to find B_n above...

IC: $u(x,0) = f(x) \Leftrightarrow \sum_{n=1}^{\infty} B_n \cdot 1 \cdot \sin\left(\frac{n\pi}{L}x\right) = f(x)$

B_n are the Fourier coeff. of $f(x)$!!

I.e. $$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

<u>Answer</u>: The sol. to the heat eq. reads

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-k\frac{n\pi}{L}t\right)\cdot\sin\left(\frac{n\pi}{L}x\right),$$ where

we don't care about convergence (here)

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \quad \text{for } n=1,2,\ldots$$

<u>Ex</u>: lecture notes $k = \frac{1}{10}$, $L = 1$, $f(x) = x(1-x)$

<u>Rem</u>:
- The above works also for other homogeneous BC : $u_x(0,t) = u_x(L,t) = 0$
- Also ok for other PDEs.

Ex: (wave equation)

Let $c>0$, $L>0$, and $f,g: \mathbb{R} \to \mathbb{R}$, consider

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 & \text{(DE)} \\ u(0,t) = u(L,t) = 0 & \text{(BC)} \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) & \text{(IC)} \end{cases}$$

(i) Separation of variable $u(x,t) = X(x) \cdot T(t)$

insert into DE:

$$X(x)T''(t) - c^2 X''(x) T(t) = 0 \iff \underbrace{\frac{T''(t)}{c^2 T(t)}}_{\text{only "t"}} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{only "x"}} = \underbrace{-\mu^2}_{\text{constant}}$$

The constant is $-\mu^2$ (see yesterday)

(ii) The above gives us two problems:

$$T''(t) = -\mu^2 c^2 T(t)$$

$$X''(x) = -\mu^2 X(x)$$

(iii) Using the BC, we get the BVP

$$\begin{cases} X''(x) + \mu^2 X(x) = 0 \\ X(0) = 0 = X(L) \end{cases}$$

(characteristic eq.: $\lambda^2 + \mu^2 \lambda = 0$ ⇝ root λ ...)

As for heat eq., we get

$$X(x) = A\cos(\mu x) + B\sin(\mu x)$$

$X(0) = 0 \Leftrightarrow X(0) = A \cdot 1 + B \cdot 0 = 0 \Rightarrow A = 0$

$X(L) = 0 \Leftrightarrow X(L) = B\sin(\mu L) = 0 \Rightarrow$

$$\mu_n = \frac{n\pi}{L} \quad \text{for } n = 1, 2, \ldots$$

$\hookrightarrow \quad X_n(x) = B_n \sin\left(\frac{n\pi}{L} x\right) \quad$ for $n = 1, 2, \ldots$

(iv) The sol. to the ODE for T is given

by $T_n(t) = A \cdot \cos(\mu_n c t) + B \sin(\mu_n c t)$

where $\mu_n = \frac{n\pi}{L}$ for $n = 1, 2, 3, \ldots$

(v) By superposition, one gets

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(c\frac{n\pi}{L}t\right) + B_n \sin\left(c\frac{n\pi}{L}t\right)\right) \sin\left(\frac{n\pi}{L}x\right)$$

and

$$u_t(x,t) = \sum_{n=1}^{\infty} \left(-A_n \frac{cn\pi}{L} \sin\left(c\frac{n\pi}{L}t\right) + B_n \frac{cn\pi}{L} \cos\left(c\frac{n\pi}{L}t\right)\right)$$

$$\cdot \sin\left(\frac{n\pi}{L}x\right)$$

(VI) Using IC, we get

$$u(x,0) = f(x) \iff \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

$$u_t(x,0) = g(x) \iff \sum_{n=1}^{\infty} \left(B_n \frac{c \cdot n \cdot \pi}{L}\right) \cdot \sin\left(\frac{n\pi}{L}x\right) = g(x)$$

These are Fourier sine series and we thus get:

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

for $n = 1, 2, \dots$

$$B_n \cdot \frac{cn\pi}{L} = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

↳ Answer:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(c\frac{n\pi}{L}t\right) + B_n \sin\left(c\frac{n\pi}{L}t\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

where A_n, B_n are given above.

3) Inhomogeneous equations:

Idea: Use superposition principle to split the PDE into 2 "easier" problems.

Ex: Solve the heat eq.

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \boxed{8} \\ u(0,t) = \boxed{1},\ u(1,t) = \boxed{3} \\ u(x,0) = f(x), \end{cases} \qquad (IC)$$

where f is given

Idea: Write $u(x,t) = v(x,t) + s(x)$,

where

$v(x,t)$ is called a transiant sol.

(takes care IC and $\xrightarrow[t\to\infty]{} 0$)

$S(x)$ is calle a steady state sol.

(s does not depend on time,

takes care of inhomogeneity)

(i) Insert ansatz $u(x,t) = v(x,t) + s(x)$ into DE:

sol. hom. prob.

particular sol.

$$V_t(x,t) - V_{xx}(x,t) - \underline{s''(x)} = \underline{8}$$

Set $-s''(x) = 8$!

$$V_t(x,t) - V_{xx}(x,t) = 0 \; !$$

(ii) Next, use BC:

$u(0,t) = 1 \iff v(0,t) + \underline{s(0)} = \underline{1}$

$u(1,t) = 3 \Leftrightarrow v(1,t) + s(1) = 3$

Set $s(0) = 1, \ s(1) = 3$

$v(0,t) = 0 = v(1,t)$

$y'' - 9y' + 4y = 8$

$\rightarrow$ solve homog. eq. $y'' - 9y' + 4y = 0$

$\hookrightarrow$ particular sol. $y_p'' - 9y_p' + 4y_p = 8$

(iii) Using IC, we get

$u(x,0) = f(x) \Leftrightarrow v(x,0) + s(x) = f(x)$ or

$v(x,0) = f(x) - s(x)$

(iv) Collecting all the above, we

find 2 "easier" problems!

$$\begin{cases} v_t(x,t) - v_{xx}(x,t) = 0 \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = f(x) - s(x) \end{cases}$$

linear homogeneous

heat eq. for v!!

"No problemo"

see previous section

and

$$\begin{cases} -s''(x) = 8 \\ s(0) = 1, \ s(1) = 3 \end{cases}$$

BVP for s!!

Integrate 2 times:

$S(x) = Ax^2 + Bx + C$

$S'(x) = 2Ax + B$

$S''(x) = 2A \overset{!}{=} -8 \Rightarrow A = -4$

$S(0) = 1 = A \cdot 0 + B \cdot 0 + C \Rightarrow C = 1$

$S(1) = 3 = A + B + C = 3 \Rightarrow B = 6$

(L) $S(x) = -4x^2 + 6x + 1$

Inserting this $S(x)$ into the IC for the linear eq. for v:

$$\begin{cases} v_t - v_{xx} = 0 \\ v(0,t) = v(1,t) = 0 \\ v(x,0) = f(x) - S(x) = f(x) - (-4x^2 + 6x + 1) \end{cases}$$

and using the previous section to find $v(x,t)$

provides us with the answer

$$u(x,t) = v(x,t) + S(x) =$$

$$v(x,t) + (-4x^2 + 6x + 1)$$